TREATISE

ON

THE ELEMENTS

OF

ALGEBRA.

BY THE LATE

REV. B. BRIDGE, B.D. F.R.S.

FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE.

ELEVENTH EDITION.

LONDON:
PRINTED FOR T. CADELL, STRAND;
DEIGHTONS, AND STEVENSON, CAMBRIDGE;
AND PARKER, OXFORD.

1842.

LONDON:
PRINTED BY RICHARD WATTS, CROWN COURT, TEMPLE BAR.

CONTENTS.

SECT. INTRODUCTION. PAGE

I. *Explanation of the Algebraic method of notation* - 1

II. *Exemplification of the Algebraic signs and symbols* - 4

CHAP. I.

On the Addition, Subtraction, Multiplication, and Division of Algebraic Quantities.

III. *Addition* - - - - - - - - 7

IV. *Subtraction* - - - - - - - - 9

V. *Multiplication* - - - - - - - 11

VI. *Division* - - - - - - - - 15

VII. *On the application of the foregoing rules to quantities with literal coefficients* - - - - - 21

VIII. *Some general theorems, deduced by means of the foregoing rules* - - - - - - - 23

CHAP. II.

On Algebraic Fractions.

IX. *On the reduction of fractions* - - - - - 25

X. *On the addition, subtraction, multiplication, and division of fractions* - - - - - - 31

XI. *On the method of finding the greatest common measure of two or more quantities* - - - - 36

CHAP. III.

On the Involution and Evolution of Numbers and of Algebraic Quantities.

XII. *On the involution of numbers and simple algebraic quantities* - - - - - - - - 41

XIII. *On the involution of compound algebraic quantities,* 42

SECT. PAGE
XIV. *On the evolution of algebraic quantities* - - 48
XV. *On the investigation of the rules for the extraction of the square and cube roots of numbers* - - 52
XVI. *On the general mode of expressing the powers and roots of quantities by means of indices* - - 57

Chap. IV.

On Simple Equations.

XVII. *On the solution of simple equations containing only one unknown quantity* - - - - - 60
XVIII. *On the solution of simple equations containing two or more unknown quantities* - - - 69
XIX. *The solution of questions producing simple equations*, 75

Chap. V.

On Quadratic Equations.

XX. *On the solution of pure quadratic equations* - - 88
XXI. *On the solution of adfected quadratic equations* - 90
XXII. *On the solution of questions producing quadratic equations* - - - - - - - 95
XXIII. *On quadratic equations having impossible roots* - 100
XXIV. *On the solution of quadratic equations of the form* $x^{2n} + px^{n} = q$ - - - - - - 101
XXV. *On the solution of quadratic equations containing two unknown quantities* - - - - - 103
XXVI. *On the solution of certain equations, in which the two unknown quantities (x and y) are similarly involved* - - - - - - - - 108

Chap. VI.

On Ratios, Proportion, and Variation.

XXVII. *Definitions* - - - - - - - - 111
XXVIII. *On the comparison and composition of ratios* - 141
XXIX. *On proportion* - - - - - - - 117
XXX. *On variation* - - - - - - - 124

CHAP. VII.

On Arithmetical and Geometrical Progression.

SECT. PAGE
XXXI. *Definitions* - - - - - - - - 130
XXXII. *On arithmetical progression* - - - - 131
XXXIII. *On geometrical progression* - - - - 136
XXXIV. *On the method of finding any number of arithmetic or geometric means between two numbers,* 138
XXXV. *On the solution of equations relating to numbers in arithmetical or geometrical progression* - 140
XXXVI. *On the summation of an infinite series of fractions in geometric progression; and on the method of finding the value of circulating decimals* - - 144

CHAP. VIII.

On Surds.

XXXVII. *On the reduction of surds* - - - - 147
XXXVIII. *On the application of the fundamental rules of arithmetic to surd quantities* - - - 151
XXXIX. *On the method of finding multipliers which shall render binomial surd quantities* rational - - 154
XL. *On the method of extracting the square root of binomial surds* - 157

CHAP. IX.

On Miscellaneous Subjects.

XLI. *On prime numbers and their relations; and on the method of finding the least common multiple of two or more numbers* - - - 160
XLII. *Properties of numbers* - - - - 163
XLIII. *Permutations and combinations* - - - 166
XLIV. *Unlimited Problems* - - - - - 168
XLV. *Diophantine Problems* - - - - 171
XLVI. *The solution of two questions relating to numbers in geometrical progression* - - - 173

CHAP. X.

On the Binomial Theorem, and subjects connected with it.

SECT. PAGE

XLVII. *The general demonstration of this Theorem* - - 176

XLVIII. *Some observations arising out of the foregoing theorem* - - - - - - - - 179

XLIX. *On the expansion of series* - - - - 181

L. *On the method of finding the approximate ratio of the powers and roots of numbers whose difference is small* - - - - - 187

LI. *On the method of extracting the* nth *root of a binomial quadratic surd* - - - - 188

LII. *On the method of reverting a series* - - 194

CHAP. XI.

On Logarithms, and subjects connected with them.

LIII. *Definition and properties of logarithms* - - 196

LIV. *On the method of finding the logarithm of any given number* - - - - - - 198

LV. *On the method of constructing logarithmic tables*, 200

LVI. *On the application of logarithms to complex arithmetical operations, and to the solution of exponential equations* - - - - - - 206

LVII. *On the summation of geometric series* - - 208

LVIII. *On compound interest* - - - - - - 211

LIX. *On the method of finding the increase of population in any Country, under given circumstances of births and mortality* - - - - 215

LX. *A Table, exhibiting the period in which the population of a Country has a tendency to* DOUBLE *itself, from an estimate of its increase* per cent. *taken at the end of every ten years* - - 221

PUBLIC
LIBRARY

ELEMENTS
OF
ALGEBRA.

INTRODUCTION.

ALGEBRA is that branch of Mathematical science in which number or quantity in general, and its several relations, are made the subject of calculation, by means of certain signs and symbols, the nature and meaning of which may be explained as follows.

I.

Explanation of the Algebraic Method of Notation.

1. Quantities whose values are *known* or *determined*, are generally expressed by the *first* letters of the Alphabet, a, b, c, d, &c.; and *unknown* or *undetermined* quantities are commonly represented by the last letters of the Alphabet, x, y, z, &c.

2. The *multiples* of these quantities, such as, *twice* a, *three times* b, *five times* x, &c. are expressed by placing *numbers* before them thus, $2\,a$, $3\,b$, $5\,x$, &c.; and the numbers 2, 3, 5, &c. thus prefixed are called the *coefficients of* a, b, x, &c. in the several quantities $2\,a$, $3\,b$, $5\,x$, &c.

3. The sign + (*plus*) placed between two or more quantities means that those quantities should be *added* together; thus, $a + b + x +$ &c. means the *sum* of the quantities a, b, x, &c.; and the sign − (*minus*) placed before any quantity means that such quantity should be *subtracted* from the

quantity or quantities with which it is combined; thus, $a-b$ means the *difference* between a and b; and $a+b-c$, the difference between $a+b$ and c.

4. In the general expression $a+2b-4x+3y-5z$, &c. such quantities as have the sign $+$ prefixed to them are called *positive* or *affirmative* quantities; and such as have the sign $-$ prefixed to them, are called *negative quantities*. If no sign be prefixed to a quantity, then the sign $+$ is understood; thus in the foregoing expression the *positive* quantities are a, $+2b$, $+3y$, and the *negative* ones $-4x$, $-5z$.

5. The general sign for the *multiplication* of quantities is $\times$; but the manner of expressing the product of two or more quantities is varied according to circumstances. The product of quantities consisting of single letters is expressed by placing those letters one after another, and generally according to the order in which they stand in the Alphabet; thus, the product of a and b is expressed by ab; of a, b, and x, by abx; of $3a$, x, and y, by $3axy$; &c. &c. The product of $a+b$ and $c+d$ is expressed by $\overline{a+b}\times\overline{c+d}$, or $\overline{a+b}\,.\,\overline{c+d}$, or $(a+b)(c+d)$; in the two former cases, the line drawn over $a+b$ and $c+d$, to mark them as distinct quantities, is called a *vinculum*.

6. The sign $\div$ placed between two quantities means that the former of those quantities is to be *divided* by the latter; thus, $a\div b$ means that a is to be divided by b; $a+b\div c+d$, that $a+b$ is to be divided by $c+d$. But since every fraction represents the quotient of the numerator divided by the denominator, this division is more simply expressed by making the former quantity the *numerator*, and the latter the *denominator* of a fraction; thus, $\frac{a}{b}$ expresses the quotient of a divided by b; and $\frac{a+b}{c+d}$, the quotient of $a+b$ by $c+d$.

7. The *powers* of algebraic quantities are expressed by placing a *small figure* (equivalent to the number of factors, and called the *index* or *exponent* of the power) at the right-hand of the letter; thus,

$a\times a$ . . . . . . or the *square* of a . . is expressed by a^2,
$b\times b\times b$ . . . . or the *cube* of b . . . . by b^3,
$x\times x\times x\times x$ . . or the *fourth power* of x . . . . . . by x^4,
$(a+b)(a+b)(a+b)$ or the *cube* of $a+b$ . . . . . . by $\overline{a+b}^{\,3}$,
and so on.

8. The *roots* of quantities are expressed by the sign $\sqrt{}$, with the proper index annexed; thus,

$\sqrt[2]{a}$, or $\sqrt{a}$, expresses the *square root* of a,
$\sqrt[3]{b}$ . . . . *cube root* of b,
$\sqrt[4]{a+x}$ . . . . . *fourth* or *biquadrate root* of $a+x$,

and so on. The roots of quantities may also be expressed by *fractional indices*; but this method of notation requires an explanation, which will be given in Chap. III.

9. *Like* quantities are such as consist of the *same letter*, or the *same combination of letters;* thus, $5a$ and $7a$; $4ab$ and $9ab$; $2bx^2$ and $6bx^2$; &c. are called *like* quantities; and *unlike* quantities are such as consist of *different letters*, or of *different combinations of letters*; thus, $4a$, $3b$, $7ax$, $5bx^2$, &c. are *unlike* quantities.

10. Algebraic quantities have also different denominations, according to the number of terms (connected by the signs $+$ or $-$) of which they consist; thus,

a, $2b$, $3ax$, &c. quantities consisting of *one* term, are called *simple* quantities.

$a+x$, a quantity consisting of *two* terms, is called a *binomial.*

$b-c$ (that particular species of binomial which expresses the *difference* between two quantities) is called a *residual.*

$bx+y-z$, a quantity consisting of *three* terms, is called a *trinomial.*

$a^2x+by-3c+d$, a quantity consisting of *four* terms, is called a *quadrinomial.*

$a+b-c+x-y$ &c. a quantity consisting of an indefinite number of terms, a *multinomial.*

11. The sign = placed between two or more quantities, expresses the *equality* of such quantities; thus, "$a+b=c+d$," means that $a+b$ is equal to $c+d$; and "$ax+by=cx+dy=ex+fy$," mean that the quantities $ax+by$, $cx+dy$, and $ex+fy$, are all equal to each other. When quantities are thus connected together by this sign of equality, the expression is called an *equation.*

12. In algebraical operations, the word *therefore,* or *consequently,* often occurs. To express this word, the symbol $\therefore$ is generally made use of; thus, the sentence "therefore $a+b$ is equal to $c+d$," is expressed by "$\therefore a+b=c+d$."

II.

Exemplification of the Algebraic Signs and Symbols.

13. The use of these several *signs, symbols,* and *abbreviations,* may be exemplified in the following manner:

Ex. 1. In the algebraic expression $a+b-c$, let $a=9$, $b=7$, and $c=3$; then

$$\begin{aligned} a+b-c &= 9+7-3 \\ &= 16-3=13. \end{aligned}$$

Ex. 2. In the expression $ax+ay-xy$, let $a=5$, $x=2$, $y=7$; then, to find its value, we have

$$\begin{aligned} ax+ay-xy &= 5\times 2+5\times 7-2\times 7 \\ &= 10+35-14 \\ &= 45-14=31. \end{aligned}$$

Ex. 3. What is the value of $\frac{ax+by}{b+x}$, where $a=5$, $b=3$, $x=7$, and $y=5$?

$$\text{Here } ax+by=5\times 7+3\times 5=35+15=50,$$
$$\text{and } b+x=3+7=10;$$
$$\therefore \frac{ax+by}{b+x}=\frac{50}{10}=5.$$

Ex. 4. In the expression $\frac{ax^2+b^2}{bx-a^2-c}$, let $a=3$, $b=5$, $c=2$, $x=6$; What is its numerical value?

$$\text{Here } ax^2+b^2=3\times 6\times 6+5\times 5=108+25=133,$$
$$\text{and } bx-a^2-c=5\times 6-3\times 3-2=30-9-2=19;$$
$$\frac{ax^2+b^2}{bx-a^2-c}=\frac{133}{19}=7.$$

Ex. 5. There is a certain algebraic expression consisting of six *terms* connected together by the sign *plus*; the *first* term of it arises from *multiplying* three times the *square* of a by the quantity b; the *second* term is the ***sum of the squares*** of a and b *divided* by the quantity c: the ***third*** is the *product* of a, b, and c; the *fourth* is ***two-thirds*** of the *product* of a and b; the *fifth* arises from ***dividing*** the ***square of*** a by the *cube of* b; and the *last* term is a fraction, whose *binomial* numerator is the *difference* between a and b, and whose *trinomial* denominator is the sum of the ***cubes*** of a and b and the *fourth power* of c.

All this is expressed, *in one line of algebraic writing*, thus;

$$3a^2b+\frac{a^2+b^2}{c}+abc+\frac{2ab}{3}+\frac{a^2}{b^3}+\frac{a-b}{a^3+b^3+c^4}.$$

Let $\left.\begin{array}{l} a=4, \\ b=3, \\ c=2; \end{array}\right\}$ then the value of this quantity is,

$$144+\frac{16+9}{2}+24+8+\frac{16}{27}+\frac{4-3}{64+27+16},$$

or

$$176+\frac{25}{2}+\frac{16}{27}+\frac{1}{107}=189\,\frac{589}{5778}.$$

CHAP. I.

ON THE ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF ALGEBRAIC QUANTITIES.

---

14. **Previously** to the application of the fundamental rules of Arithmetic to Algebraic quantities, it may be proper to observe, that, although the explanation of the sign *minus* in Art. 3. does not, in strictness, extend beyond the subtraction of a less quantity from a greater one, it is convenient to consider negative quantities abstractedly, without any reference to others from which they may be supposed to be subtracted. For although, when we say that $2-5$ is equal to -3, we mean nothing more than that the addition of 2 and subtraction of 5, is, on the whole, equivalent to the subtraction of 3; yet, after the algebraic operation has been performed upon it, the quantity $2-5$ assumes the definite value of -3.

It must be farther observed, that the word Addition is, in Algebra, taken in a much more comprehensive sense than in common Arithmetic; and as denoting the *union* of two or more quantities, *positive* or *negative*. Thus, the union of 2 with -5, in the foregoing example, is called the *addition* of those quantities. The same remark is to be extended to Subtraction; which is, properly, the finding such a quantity, as, being *algebraically* added to the subtrahend, will give the quantity from which the subtraction is made.

III.

ADDITION.

From the division of algebraic quantities into *positive* and *negative*, *like* and *unlike*, there arise three cases of Addition.

CASE I.

To add like quantities with like signs.

15. In this case, the rule is, "To add the coefficients of "the several quantities together, and to the result annex "the common sign, and the common letter or letters;" for it is evident, from the common principles of Arithmetic, if $+2a$, $+3a$, and $+5a$ be added together, their sum must be $+10a$; and if $-3b^2$, $-4b^2$, and $-8b^2$ be added together, their sum must be $-15b^2$.

$$\begin{array}{rrr}
\text{Ex. 1.} & \text{Ex. 2.} & \text{Ex. 3.}\\
2x+3a-4b & 7x^2+3xy-5bc & 4a^3-3a^2+1\\
3x+2a-5b & 9x^2+2xy-7bc & 2a^3-a^2+17\\
4x+8a-7b & 11x^2+5xy-4bc & 5a^3-2a^2+4\\
9x+4a-6b & ^{(a)}\,x^2+4xy-bc & 3a^3-7a^2+3\\
5x+7a-9b & x^2+9xy-2bc & a^3-a^2+10\\
\hline
23x+24a-31b & 29x^2+23xy-19bc & 15a^3-14a^2+35\\
\hline\hline
\end{array}$$

$$\begin{array}{rrr}
\text{Ex. 4.} & \text{Ex. 5.} & \text{Ex. 6.}\\
3x^3+4x^2-x & 7a^3-3a^2b+2ab^2-3b^3 & 2x^2y-3x+2\\
2x^3+x^2-3x & 4a^3-a^2b+ab^2-b^3 & 4x^2y-2x+1\\
7x^3+2x^2-2x & a^3-2a^2b+3ab^2-5b^3 & 3x^2y-5x+10\\
4x^3+x^2-x & 5a^3-3a^2b+4ab^2-2b^3 & x^2y-x+15\\
\hline
 & & \\
\hline\hline
\end{array}$$

(a) In these Examples, it may be observed that some of the quantities have *no coefficient*. In this case, *unity* or 1 is *always understood*. Thus, in adding up this column, we say, $1+1+11$[illegible]$+7=29$; in the *third*, $2+1+4+7+5=19$; and so of the rest.

Case II.

To add like quantities with unlike signs.

16. Since the compound quantity $a+b-c+d-e$ &c. is positive or negative, according as the sum of the positive terms is greater or less than the sum of the negative ones, the aggregate or sum of the quantities $2a-4a+7a-3a$ will be $+2a$, and that of the quantities $7b^2-5b^2+2b^2-8b^2$ will be $-4b^2$; for in the former case, the excess of the sum of the positive terms above the negative ones is $2a$; and in the latter, that of the negative above the positive is $4b^2$. Hence this general rule for the addition of like quantities with unlike signs; "Collect the coefficients of the *positive* "terms into one sum, and also those of the *negative*; sub-"tract the *lesser* of these sums from the *greater*; to this "*difference*, annex the sign of the *greater* together with "the common letter or letters, and the result will be the "sum required."

If the aggregate of the positive terms be *equal* to that of the negative ones, then this *difference* is equal to 0; and consequently the sum of the quantities will be equal to 0, as in the *second* column of Ex. 2. following.

Ex. 1.	Ex. 2.	Ex. 3.
$4x^2-3x+4$	$-7ab+3bc-xy$	$-5x^3+13x$
$-2x^2+x-5$	$-ab+2bc+4xy$	$-2x^3-4x$
$3x^2-5x+1$	$3ab-bc+2xy$	$7x^3+x$
$7x^2+2x-4$	$-2ab+4bc-3xy$	$9x^3-14x$
$-x^2-4x+13$	$5ab-8bc+xy$	$-13x^3-2x$
$11x^2-9x+9$	$-2ab$ * $+3xy$	$-4x^3-6x$

Ex. 4.	Ex. 5.	Ex. 6.
$4x^3-2x+3y$	$5a^3-2ab+b^2$	$4x^2y^2-2xy-3$
$-x^3+4x-y$	$-a^3+ab+2b^2$	$-x^2y^2-xy-1$
$7x^3-x+9y$	$4a^3-3ab+b^2$	$3x^2y^2+4xy-5$
$9x^3+21x-2y$	$2a^3+4ab-4b^2$	$-9x^2y^2-2xy+9$
$19x^3+22x+9y$	[illegible] 0 0	$-3x^2y^2-xy$ 0

CASE

CASE III.

17. There now only remains the case where *unlike* quantities are to be added together, which must be done by collecting them together into one line, and annexing their proper signs; thus the sum of $3x, -2a, +5b, -4y$, is $3x-2a+5b-4y$; except when *like* and *unlike* quantities are mixed together, as in the following examples, where the expressions may be simplified, by collecting together such quantities as will coalesce into one sum.

Ex. 1.

$$\begin{array}{l}
3ab + x - y \\
4c - 2y + x^2 \\
5ab - 3c + d \\
4y + x - 2y \\
\hline
8ab + 2x - y + c + d + x^2
\end{array}$$

Collecting together *like* quantities, and beginning with $3ab$, we have $3ab + 5ab = 8ab$; $+x + x = +2x$; $-y - 2y + 4y - 2y = -y$; $4c - 3c = +c$; besides which there are the two quantities $+d$ and $+x^2$, which do not coalesce with any of the others; the sum required therefore is $8ab + 2x - y + c + d + x^2$.

Ex. 2.

$$\begin{array}{l}
4x^2 - 2xy + 1 - 3y + 4x^3 \\
4y + 3x^3 - y^2 + xy - x^2 \\
5x^3 - 2x + y - 15 + y^2 \\
\hline
3x^2 - xy - 14 + 2y + 12x^3 - 2x
\end{array}$$

Here $4x^2 - x^2 = 3x^2$

$$\begin{aligned}
-2xy + xy &= -xy \\
+1 - 15 &= -14 \\
-3y + 4y + y &= +2y \\
+4x^3 + 3x^3 + 5x^3 &= +12x^3 \\
-y^2 + y &= 0 \\
-2x &= -2x.
\end{aligned}$$

IV.

SUBTRACTION.

18. If it were required to subtract $5-2$ (i.e. 3) from 9, it is evident that the remainder would be *greater* by 2,

ed. For the same reason, if $b-c$

were subtracted from a, the remainder would be greater by c, than if b were subtracted. Now, if b is subtracted from a, the remainder is $a-b$; and consequently, if $b-c$ be subtracted from a, the remainder will be $a-b+c$. Hence this general Rule for the subtraction of algebraic quantities; "Change the signs of the quantities *to be* "*subtracted*, and then place them one after another, as in " Addition."

Ex. 1. From $5a+3x-2b$, take $2c-4y$. The quantity to be subtracted *with its signs changed*, is $-2c+4y$; therefore the remainder is $5a+3x-2b-2c+4y$.

Ex. 2. From $7x^2-2x+5$, take $3x^2+5x-1$.
The remainder is $7x^2-2x+5-3x^2-5x+1$,
or $7x^2-3x^2-2x-5x+5+1=4x^2-7x+6$.

But when *like* quantities are to be subtracted from each other, as in Ex. 2, the better way is to set one row under the other, and apply the following Rule; "*Conceive* the "signs of the *quantities to be subtracted* to be *changed*, and " then proceed as in Addition."

	Ex. 3.	Ex. 4.	Ex. 5.
From	$7x^2-2x+5$	$12a^2-3a+b-1$	$5y^2-4y+3a$
Subtract	$3x^2+5x-1$	$6a^2+a-2b+3$	$6y^2-4y-a$
Remainder	$4x^2-7x+6$	$6a^2-4a+3b-4$	$-y^2$ * $+4a$

	Ex. 6.	Ex. 7.	Ex. 8.
From	$7xy+2x-3y$	$14x+y-z-5$	$13x^3-2x^2+7$
Subtract	$2xy-x+y$	$x+y+z-11$	$-x^3+x^2-6$
Remainde		[illegible]	[illegible]

V.

MULTIPLICATION.

19. In the multiplication of algebraic quantities, the four following Rules must be observed.

I. When quantities having *like* signs are multiplied together, the sign of the *product* will be + ; and if their signs are *unlike*, the sign of the *product* will be —.*

II. The coefficients of the *factors* must be multiplied together, to form the coëfficient of the *product*.

---

* This Rule for the multiplication of the Signs may be thus explained;

To multiply $a-b$ by $c-d$, is to add $a-b$ to itself as often as there are units in $c-d$; now this is done by *adding* it *c times*, and *subtracting* it *d times*;

But $a-b$, *added c times* . . $= ac-bc$,

and $a-b$, *subtracted d times* $= -ad+bd$,

$\therefore \overline{a-b}\times\overline{c-d}$ . . . . . . $= ac-bc-ad+bd$.

i. e. $+a\times+c=+ac$

$-b\times+c=-bc$

$+a\times-d=-ad$

$-b\times-d=+bd$.

Or thus;

I. If $+a$ is to be multiplied by $+b$, it means, that $+a$ is to be *added* to itself as often as there are units in b; and, consequently, the product will be $+ab$.

II. If $-a$ is to be multiplied by $+b$, it means, that $-a$ is to be *added* to itself as often as there [illegible] units in b [illegible] therefore [illegible] product is $-ab$.

III. If $+a$ is to be multiplied by $-b$, it means, that $+a$ is to be *subtracted* as often as there are units in b, as appears from the foregoing explanation; and consequently the product is $-ab$.

IV. If $-a$ is to be multiplied by $-b$, it means that $-a$ is to be *subtracted* as often as there are units in b; and, since to *subtract a negative quantity* is the same as *to add a positive one*, the product will be $+ab$.

III. The letters of which they are composed must be set down, one after another; and generally *according to their order in the Alphabet.*

IV. If the *same* letter is found in both factors, the indices of it must be *added* together, to form the index of it in the *product.* This follows immediately from Art. 7, as will appear by the following example;

$$a^3 \times a^2 = aaa \times aa = aaaaa = a^5.$$

Thus, $+a$ multiplied by $+b$ is equal to $+ab$, and $-a$ multiplied by $-b$ is also equal to $+ab$; $+3x \times -5y = -15xy$; $-3ab \times +4cd = -12abcd$; $-4a^2b^2 \times -3abd^2 = +12a^3b^3d^2$; &c. &c.

From the division of algebraic quantities into *simple* and *compound,* there arise three cases of Multiplication. In performing the operation, the Rule is, "To determine *first* the sign, *then* the coefficient, and *afterwards* the letters."

CASE I.

20. When *both* factors are *simple* quantities; for which the Rule has been already given.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$4ab$	$2axy$	$-3abc$	$-5a^2bc$
$3a$	$-3y$	$5a^2b$	$-2b^2x^2$
$12a^2b$	$-6axy^2$	$-15a^3b^2c$	$+10a^2b^3cx^2$

Ex. 5.	Ex. 6.	Ex. 7.	Ex. 8.
$4abc$	$9x^2y^2$	$-4cdx$	$-7ax^2y$
$3ac$	$-2y$	$-2c$	$-2ac^2x$
[illegible]	[illegible]	[illegible]	

CASE II.

21. When one factor is *compound* and the other *simple;* "Then *each term* of the compound factor must be multi-

"plied by the simple factor, as in the last Case; and the "result will be the product required."

	Ex. 1.	Ex. 2.
Multiply	$3ab-2ac+d$	$3x^3-2x^2+4$
by	$4a$	$-14ax$
Product	$12a^2b-8a^2c+4ad$	$-42ax^4+28ax^3-56ax$

	Ex. 3.	Ex. 4.
Multiply	$7x^2-2x+4a$	$12a^3-2a^2+4a-1$
by	$-3a$	$3x$
Product	$-21ax^2+6ax-12a^2$	$36a^3x-6a^2x+12ax-3x$

	Ex. 5.	Ex. 6.
Multiply	$9a^2x+3a-x+1$	$4x^2y+3x-2y$
by	$-x^2$	$-3xy$
		− [illegible]

CASE III.

22. When *both* factors are *compound* quantities, each term of the multiplicand must be multiplied by each term of the multiplier; and then placing *like quantities under each other*, the sum of all the terms will be the product required.

	Ex. 1.	Ex. 2.	Ex. 3.
Multiply	$a+b$	$a+b$	a^2+ab+b^2
by	$a+b$	$a-b$	$a-b$
1st, by a . .	a^2+ab	a^2+ab	$a^3+a^2b+ab^2$
2d, by b	$ab+b^2$	$-ab-b^2$	$-a^2b-ab^2-b^3$
Product	$a^2+2ab+b^2$	$a^2\ \ *\ -b^2$	$a^3\ \ *\ \ *\ -b^3$

Ex. 4.

$$\begin{array}{r}
3x^2+\ 2x \\
4x+\ 7 \\
\hline
12x^3+\ 8x^2 \qquad\qquad \\
+21x^2+14x \\
\hline
12x^3+29x^2+14x \\
\hline\hline
\end{array}$$

Ex. 5.

$$\begin{array}{r}
3x^2-\ 2x+5 \qquad\qquad \\
6x-\ 7 \qquad\qquad \\
\hline
18x^3-12x^2+30x \qquad\quad \\
-21x^2+14x-35 \\
\hline
18x^3-33x^2+44x-35 \\
\hline\hline
\end{array}$$

Ex. 6.
$$\begin{array}{l}
14ac-\ 3ab\ +\ 2 \\
\quad ac-\quad ab+\ 1 \\
\hline
14a^2c^2-\ 3a^2bc+\ 2ac \\
\qquad -14a^2bc \qquad\qquad +3a^2b^2-2ab \\
\qquad\qquad\qquad +14ac \qquad\qquad -3ab+2 \\
\hline
14a^2c^2-17a^2bc+16ac+3a^2b^2-5ab+2
\end{array}$$

[illegible]

Ex. 7.
$$\begin{array}{l}
x^2-\frac{1}{2}x+\frac{2}{3} \\
\frac{1}{3}x+ \\
\hline
\frac{1}{3}x^3-\frac{1}{6}x^2+\frac{2}{9}x \\
\qquad +\ 2x^2-\quad x+\frac{4}{3} \\
\hline
\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3}
\end{array}$$

[illegible]

Ex. 8. Multiply $a^3+3a^2b+3ab^2+b^3$ . . by $a+b$.

ANSWER, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

Ex. 9. . . . . . . $4x^2y+3xy-1$ . . . . . by $2x^2-x$.

ANSW. $8x^4y+2x^3y-2x^2-3x^2y+x$.

Ex. 10. . . . . . x^3-x^2+x-5 . . . . . by $2x^2+x+1$.

ANSW. $2x^5-x^4+2x^3-10x^2-4x-5$.

Ex. 11. . . . . . $3a^2+2ab-b^2$ . . . . . by $3a^2-2ab+b^2$.

ANSW. $9a^4-4a^2b^2+4ab^3-b^4$.

Ex. 12. Multiply $x^3+x^2y+xy^2+y^3$ . . by $x-y$.

Answ. x^4-y^4.

Ex. 13. . . . . . $x^2-\frac{3}{4}x+1$ . . . . . . by $x^2-\frac{1}{2}x$.

Answ. $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$.

VI.

DIVISION.

23. In the division of Algebraic quantities, the four following Rules (which arise immediately out of the consideration that the quotient multiplied by the divisor gives the dividend) are to be observed.

I. That if the signs of the dividend and divisor be *like*, then the sign of the quotient will be $+$; if *unlike*, then the sign of the quotient will be $-$.(a)

II. That the coefficient of the *dividend* is to be divided by the coefficient of the *divisor*, to obtain the coefficient of the quotient.

III. That all the letters *common* to both the dividend and the divisor must be *rejected* in the quotient.(b)

(a) The Rule for the *signs* follows immediately from that in Multiplication; thus,

Since $+a\times+b=+ab$, $\therefore\frac{+ab}{+a}=+b$, and $\frac{+ab}{+b}=+a$ i.e. *like* signs

$+a\times-b=-ab$, $\therefore\frac{-ab}{+a}=-b$, and $\frac{-ab}{-b}=+a$ produce $+$,

$-a\times-b=+ab$, $\therefore\frac{+ab}{-a}=-b$, and $\frac{+ab}{-b}=-a$ and *unlike* signs $-$.

(b) If any letter or letters are found in the divisor, which are not in the dividend, they must remain in the denominator of the fraction by which the division is expressed. See Art. 35, with which this case coincides, and the examples there.

IV. That if the same letter be found in both the dividend and divisor with *different* indices, then the index of that letter in the divisor must be *subtracted* from its index in the dividend, to obtain its index in the quotient. Thus,

I. $+abc$ divided by $+ac$ . . . or $\frac{+abc}{+ac} = +b$.

II. $+6abc$ . . . . . . $-2a$ . . . or $\frac{6abc}{-2a} = -3bc$.

III. $-10xyz$ . . . . . $+5y$ . . . or $\frac{-10xyz}{+5y} = -2xz$.

IV. $-20a^2x^2y^3$ . . . . $-4axy$, or $\frac{-20a^2x^2y^3}{-4axy} = +5axy^2$.(a)

Of *Division*, also, there are three Cases; the same as in *Multiplication*.

CASE I.

24. When the dividend and divisor are both *simple* terms.

Ex. 1.

Divide $18ax^2$ by $3ax$.

$\frac{18ax^2}{3ax} = 6x$.

Ex. 2.

Divide $15a\,b^2$ by $-5a$.

$\frac{+15a^2b}{-5a} = -3ab$.

Ex. 3.

Divide $-28x^2y^3$ by $-4xy$.

$\frac{-28x^2y^3}{-4xy} = +7xy^2$.

Ex. 4.

Divide $25a^3c^4$ by $-5a^2c$.

$\frac{+25a^3c^4}{-5a^2c} =$

Ex. 5.

Divide $-14a^3b^2c$ by $7ac$.

$\frac{-14a^3b^2c}{+7ac} =$

Ex. 6.

Divide $-20x^2y^2z^3$ by $-4yz$.

$\frac{-20x^2y^2z^3}{-4yz} =$

CASE II.

(a) If the index of any letter in the divisor should be greater than that of the same letter in the dividend, the index in the quotient will, by the Rule, be negative. The signification of this negative index will be explained in Art. 66.

CASE II.

25. When the dividend is a *compound* quantity, and the divisor a *simple* one, then each term of the dividend must be divided separately, and the resulting quantities will be the quotient required.

Ex. 1. Divide $42a+3ab+12a^2$ by $3a$.

$$\frac{42a+3ab+12a^2}{3a}=14+b+4a.$$

Ex. 2. Divide $90a^2x^3-18ax^2+4a^2x-2ax$ by $2ax$.

$$\frac{90a^2x^3-18ax^2+4a^2x-2ax}{2ax}=45ax^2-9x+2a-1.$$

Ex. 3. Divide $4x^3-2x^2+2x$ by $2x$.

$$\frac{4x^3-2x^2+2x}{2x}=$$

Ex. 4. Divide $-24a^2x^2y-3axy+6x^2y^2$ by $-3xy$.

$$\frac{-24a^2x^2y-3axy+6x^2y^2}{-3xy}=$$

Ex. 5. Divide $14ab^4+7a^2b^2-21a^2b^3+35a^3b$ by $7ab$.

$$\frac{14ab^4+7a^2b^2-21a^2b^3+35a^3b}{7ab}=$$

CASE III.

26. When the dividend and divisor are *both compound* quantities. In this case, the Rule is, "to arrange both dividend and divisor according to the powers of the same "letter, beginning with the *highest;* then find how often "the first term of the divisor is contained in the first term "of the dividend, and place the result in the quotient; multiply each term of the divisor by this quantity, and subtract the product from the dividend; to the remainder "bring down as many terms of the dividend, as will make its

" number of terms equal to the number of those in the di-
" visor; and then proceed as before, till all the terms of the
" dividend are brought down, as in common arithmetic."

Ex. 1.

Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

$$\begin{array}{r|l}
a-b & a^3-3a^2b+3ab^2-b^3\,(a^2-2ab+b^2 \\
 & a^3-a^2b \\ \hline
 & *\ -2a^2b+3ab^2 \\
 & -2a^2b+2ab^2 \\ \hline
 & \ *\quad ab^2-b^3 \\
 & ab^2-b^3 \\ \hline
 & *\quad *
\end{array}$$

In this Example, the dividend is arranged according to the powers of a, the first term of the divisor. Having done this, we proceed by the following steps;

I. a is contained in a^3, a^2 times; put this in the quotient.

II. Multiply $a-b$ by a^2, and it gives a^3-a^2b.

III. Subtract a^3-a^2b from a^3-3a^2b, and the remainder is $-2a^2b$.

IV. Bring down the next term $+3ab^2$.

V. a is contained in $-2a^2b$, $-2ab$ times; put this in the quotient.

VI. *Multiply* and *subtract* as before, and the remainder is ab^2.

VII. Bring down the last term $-b^3$.

VIII. a is contained in ab^2, $+b^2$ times; put this in the quotient.

IX. *Multiply* and *subtract* as before, and nothing remains; the *quotient* therefore is $a^2-2ab+b^2$.

Ex. 2.

Ex. 2.

$$\begin{array}{l}
a^2+2ax+x^2)\,a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5\,(a^3+3a^2x+3ax^2+x^3\\
\qquad a^5+2a^4x+a^3x^2\\
\quad * \qquad 3a^4x+9a^3x^2+10a^2x^3\\
\qquad\qquad 3a^4x+6a^3x^2+3a^2x^3\\
\qquad\qquad\qquad 3a^3x^2+7a^2x^3+5ax^4\\
\qquad\qquad\qquad 3a^3x^2+6a^2x^3+3ax^4\\
\qquad\qquad\qquad * \qquad a^2x^3+2ax^4+x^5\\
\qquad\qquad\qquad\qquad\quad a^2x^3+2ax^4+x^5
\end{array}$$

Ex. 3.

$$\begin{array}{l}
4x^2-7x)\,12x^5-13x^4-34x^3+40x^2\,(3x^3+2x^2-5x+\dfrac{5x^2}{4x^2-7x}\,(a)\\
\quad * \quad 12x^5-21x^4\\
\qquad\qquad +8x^4-34x^3\\
\qquad\qquad +8x^4-14x^3\\
\qquad\qquad\quad * \quad -20x^3+40x^2\\
\qquad\qquad\qquad -20x^3+35x^2\\
\qquad\qquad\qquad * \quad +5x^2
\end{array}$$

Ex. 4.

$$\begin{array}{l}
3x-6)\,6x^4-96\,(2x^3+4x^2+8x+16\\
\qquad 6x^4-12x^3\\
\qquad * \quad +12x^3-96\\
\qquad\quad\; +12x^3-24x^2\\
\qquad\qquad * \quad +24x^2-96\\
\qquad\qquad\quad\; +24x^2-48x\\
\qquad\qquad\qquad * \quad +48x-96\\
\qquad\qquad\qquad\quad\; +48x-96\\
\qquad\qquad\qquad\qquad * \qquad *
\end{array}$$

(a) When there is a *remainder*, it must be made the *numerator* of a Fraction whose denominator is the *divisor;* this Fraction must then be placed in the *quotient* (with its proper sign), the same as in common Arithmetic.

Ex. 5.

$$x^2+x-1\,\big)\,x^6-x^4+x^3-x^2-1\,\Big(\,x^4-x^3+x^2-x+1-\frac{2x}{x^2-x+1}$$

$$\begin{array}{l}
x^6+x^5-x^4 \\
\hline
-x^5+x^3-x^2 \\
-x^5-x^4+x^3 \\
\hline
x^4-x^2-1 \\
x^4+x^3-x^2 \\
\hline
-x^3-1 \\
-x^3-x^2+x \\
\hline
x^2-x-1 \\
x^2+x-1 \\
\hline
-2x \\
\hline
\end{array}$$

Ex. 6.

$$1+x\,\big)\,1\,\Big(\,1-x+x^2-x^3+\frac{x}{1+x}$$

$$\begin{array}{l}
1+x \\
\\
-x-x^2 \\
\\
x^2 \\
x^2+x^3 \\
\hline
-x^3 \\
-x^3-x^4
\end{array}$$

In this last Example, the division may be continued to any number of terms at pleasure, observing only to place the whole divisor under the last remainder.

Ex. 7. Divide $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ by $a+b$.

Answer, $a^3+3a^2b+3ab^2+b^3$.

Ex. 8. . . . . $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^3-3a^2x+3ax^2-x^3$.

Answ. $a^2-2ax+x^2$.

Ex. 9. Divide $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$.

ANSWER, $5x^3+4x^2+3x+2$.

Ex. 10. . . . . $a^4+8a^3x+24a^2x^2+32ax^3+16x^4$ by $a+2x$.

ANSW. $a^3+6a^2x+12ax^2+8x^3$.

Ex. 11. . . . . a^5-x^5 by $a-x$.

ANSW. $a^4+a^3x+a^2x^2+ax^3+x^4$.

Ex. 12. . . . . $6x^4+9x^2-20x$ by $3x^2-3x$.

ANSW. $2x^2+2x+5-\frac{5x}{3x^2-3x}$.

Ex. 13. . . . . $9x^6-46x^5+95x^2+150x$ by x^2-4x-5.

ANSW. $9x^4-10x^3+5x^2-30x$.

Ex. 14. . . . a^2 by $1-x^3$.

ANSW. $a^2+a^2x^3+a^2x^6+\frac{a^2x^9}{1-x^3}$.

VII.

On the application of the foregoing Rules to Quantities with literal Coefficients.

27. In applying the foregoing Rules to quantities with *literal* coefficients, such as, mx, ny, qx^2 &c. (where m, n, q, &c. may be considered as the coefficients of x, y, x^2, &c.) a compound quantity may be expressed by placing the coefficients of *like* quantities one after another (with their proper signs) in a parenthesis, and then annexing the common letter or letters. Thus, the *sum* of mx and nx, which is $mx+nx$, may be expressed by $(m+n)x$; their *difference*, which is $mx-nx$, by $(m-n)x$; the multimonial $mx^2+nx^2-px^2+qx^2$, by $(m+n-p+q)x^2$; and the mixed multinomial $pxy+qy^2-rxy+my^2-nxy$, by $(p-r-n)xy+(q+m)y^2$; &c. &c. According to this method of notation the operations are performed in the following Examples.

Ex. 1.

$$\text{Add}\left\{\begin{array}{r} my^2+ny+z \\ -py^2-ry+nz \\ qy^2+my-vz \\ +ry-qz \end{array}\right.$$

$$\underline{(m-p+q)y^2+(n+m)y+(1+n-v-q)z.}$$

Ex. 2.

$$\begin{array}{ll} \text{From} & px^3+qx^2-rx+s \\ \text{Subtract} & mx^3-nx^2+tx-v \\ \text{Remainder} & \underline{(p-m)x^3+(q+n)x^2-(r+t)x+s+v.^{(a)}} \end{array}$$

Ex. 3.

$$\begin{array}{rl} \text{Multiply} & px^2+qx-r \\ \text{by} & \underline{mx-n} \\ & mpx^3+mqx^2-mrx \\ & \quad -npx^2-nqx+nr \\ \text{Product} & \underline{mpx^3+(mq-np)x^2-(mr+nq)x+nr.} \end{array}$$

Ex. 4.

$$\begin{array}{rl} \text{Multiply} & ax^2-bx+c \\ \text{by} & \underline{x^2-cx+1} \\ & ax^4-bx^3+cx^2 \\ & \quad -acx^3+bcx^2-c^2x \\ & \qquad +ax^2-bx+c \\ \text{Product} & \underline{ax^4-(b+ac)x^3+(c+bc+a)x^2-(c^2+b)x+c.} \end{array}$$

(a) As the sign prefixed to quantities in a parenthesis affects them *all*; when this sign is *negative*, the signs of all those quantities must be changed in putting them into the parenthesis. Thus, when $+tx$ is subtracted from $-rx$, the result is $-rx-tx$; and, as this means that the *sum* of rx and tx is to be *subtracted*, that *negative* sum is expressed by $-(rx+tx)=-(r+t)x$. For the same reason, any *multinomial* quantity $-mx^2-nx^2-qx^2+rx^2$, when put into a parenthesis with a *negative* sign prefixed, becomes $-(m-n+q-r)x^2$.

Ex. 5. (Division.)

$$
\begin{array}{rllll}
-cx+1\big) & ax^4 & -(b+ac)x^3 & +(c+bc+a)x^2 & -(c^2+b)x+c\big(ax^2-bx+c \\
 & ax^4 & -acx^3 & +ax^2 & \\
\hline
 & * & -bx^3+ & (c+bc)x^2 & -(c^2+b)x \\
 & & -bx^3 & +bcx^2 & -bx \\
\hline
 & & & +cx^2 & +c \\
 & & & +cx^2 & +c
\end{array}
$$

Ex. 6. Multiply mx^2-nx-r . . . by $nx-r$.

Answer, $mnx^3-(n^2+mr)x^2+r^2$.

Ex. 7. Multiply x^3-px^2+qx-r . by $x-a$.

Answ. $x^4-(a+p)x^3+(q+ap)x^2-(r+aq)x+ar$.

Ex. 8. Multiply px^2-rx+q . . . . by x^2-rx-q.

Answ. $px^4-(1+p)rx^3+(q+r^2-pq)x^2-q^2$.

Ex. 9. Divide $ax^3-(a^2+b)x^2+b^2$. by $ax-b$.

Answ. x^2-ax-b.

VIII.

Some general Theorems, deduced by means of the foregoing Rules.

From the clear and distinct manner in which quantity and its several relations are represented throughout every part of an Algebraic operation, the exemplification of its most ordinary rules affords the means of investigating certain general Theorems relating to the *sum*, *difference*, *product*, &c. &c. of numbers, of which the following are examples.

28. Let a and b be any two numbers of which a is the greater and b the lesser, and let their *sum* be represented by s and their *difference* by d,

$$
\begin{array}{rl}
\text{Then} & a+b=s \\
\text{and} & a-b=d \\
\\
\text{by Addition, } 2a & =s+d \\
\text{and } a & =\frac{s}{2}+\frac{d}{2}
\end{array}
$$

by

$$\left.\begin{array}{r}\text{by Subtraction, } 2b=s-d \\ \text{and } b=\frac{s}{2}-\frac{d}{2}\end{array}\right\}$$

From which we deduce this general Theorem, that "if the "*sum* and *difference* of any two numbers be given, the "*greater* of them may be found by adding half the given "sum to half the given difference; and the *lesser*, by subtracting half the given difference from half the given sum."

29. Let a, b, s, d have the same relation as before, then

$$s=a+b$$
$$d=a-b$$

Hence, by Multiplication, $s\times d=a^2-b^2$ (See Ex. 2. Case III. p. 14.)

$$\therefore\ s=\frac{a^2-b^2}{d}$$

$$\text{and } d=\frac{a^2-b^2}{s}.$$

From which it appears, that "if the *sum* and *difference* of "any two numbers be multiplied together, the *product* of "that sum and difference gives the *difference of the squares* "of the two numbers;" and, that "if the difference of the "squares of the two numbers be divided by their *difference*, it gives their *sum*; and if by their *sum*, it gives their "difference."

30. Let the number c be divided into any two parts a and b,

$$\text{Then } c=a+b$$
$$c=a+b$$

$\therefore$ by Multiplication, $c^2=a^2+2ab+b^2$ (See Ex. 1. Case III. p. 14.)

From which we infer, that "if a number be divided into "two parts, the *square* of the number is equal to the *sum* "*of the squares* of the two parts, together with *twice the* "*product* of those parts."

31. Let a and b be any two numbers; then,

Their difference $=a-b$

The difference of their cubes $=a^3-b^3$

By actual division, $a-b)\,a^3-b^3\,(a^2+ab+b^2$ (quotient)

$$\begin{array}{rl} & a^3-a^2b \\ \hline & +a^2b-b^3 \\ & +a^2b-ab^2 \\ \hline & \qquad +ab^2-b^3 \\ & \qquad +ab^2-b^3 \\ \hline & \qquad\quad * \quad * \end{array}$$

Hence it appears, that "if the *difference of the cubes* of "any two numbers be divided by their *difference*, the *quotient* arising will be equal to the *sum of the squares* of the "two numbers together with their *product*."

CHAP. II.

ON ALGEBRAIC FRACTIONS.

THE Rules for the management of Algebraic Fractions are the same with those in Common Arithmetic.

IX.

ON THE REDUCTION OF FRACTIONS.

32. *To reduce a Mixed Quantity to a Fraction.*

RULE. "Multiply the *integral* part by the denominator "of the *fractional*, and to the *product* annex the numerator "with its proper sign; under this *sum* place the former "denominator, and the result is the fraction required."

Ex. 1. Reduce $3a+\frac{2x}{5a^2}$ to a fraction.

The integral part × the *denominator* of the fraction + the *numerator* $=3a\times 5a^2+2x=15a^3+2x$,

Hence, $\frac{15a^3+2x}{5a^2}$ is the fraction required.

Ex. 2.

Reduce $5x-\frac{4b}{6a^2}$ to a fraction.

Here $5x\times 6a^2=30a^2x$; to this add the numerator with its proper sign, viz. $-4b$; then $\frac{30a^2x-4b}{6a^2}$. is the fraction required.

Ex. 3.

Reduce $5x-\frac{2x-3}{7}$ to a fraction.

Here $5x\times 7=35x$. In adding the numerator $2x-3$ *with its proper sign*, it is to be recollected, that the sign $-$ affixed to the fraction $\frac{2x-3}{7}$ means that the *whole* of that fraction is to be *subtracted*, and consequently the signs of each term of the numerator must be *changed* when it is combined with $35x$; hence the fraction required is $\frac{35x-2x+3}{7}=\frac{33x+3}{7}$.

Ex. 4. Reduce $4ab+\frac{2c}{3a}$ to a fraction.

Answer, $\frac{12a^2b+2c}{3a}$.

Ex. 5. . . . . . $3b^2-\frac{4a}{5x}$ to a fraction.

Answ. $\frac{15b^2x-4a}{5x}$.

Ex. 6. . . . . . $a-x+\frac{a^2-ax}{x}$ to a fraction.

Answ. $\frac{a^2-x^2}{x}$

Ex. 7. . . . . . $3x^2-\frac{4x-9}{10}$ to a fraction.

Answ. $\frac{30x^2-4x+9}{10}$.

33. *To reduce a Fraction to a Mixed Quantity.*

Rule. "Observe which terms of the *numerator* are "divisible by the *denominator* without a remainder, the "quotient will give the *integral* part; to this annex (with "their proper signs, and observing the caution given in "Ex. 3 of the last Article) the remaining terms of the "numerator with the denominator under them, and the "result will be the mixed quantity required."

Example 1.

Reduce $\frac{a^2+ab+b^2}{a}$ to a mixed quantity.

Here $\frac{a^2+ab}{a}=a+b$ is the *integral* part,

and $\frac{b^2}{a}$ is the *fractional* part;

$\therefore a+b+\frac{b^2}{a}$ is the mixed quantity required.

Ex. 2.

Reduce $\frac{15a^2+2x-3c}{5a}$ to a mixed quantity.

Here $\frac{15a^2}{5a}=3a$ is the *integral* part,

and $\frac{2x-3c}{5a}$ is the *fractional* part;

$\therefore 3a+\frac{2x-3c}{5a}$ is the mixed quantity required.

Ex. 3. Reduce $\frac{4x^2-5a}{2x}$ to a mixed quantity.

Answer, $2x-\frac{5a}{2x}$.

Ex. 4. . . . . . $\frac{12a^2+4a-3c}{4a}$ to a mixed quantity.

Answ. $3a+1-\frac{3c}{4a}$.

Ex. 5. Reduce $\frac{25x^3-3a+2c}{5x}$ to a mixed quantity.

ANSWER, $5x^2-\frac{3a-2c}{5x}$.

34. *To reduce Fractions to a common Denominator.*

RULE. "Multiply each numerator into every denominator *but its own* for the new numerators, and *all the denominators together* for the common denominators."

EXAMPLE 1.

Reduce $\frac{2x}{3}$, $\frac{5x}{b}$, and $\frac{4a}{5}$, to a common denominator.

$$\left.\begin{array}{l} 2x\times b\times 5=10bx \\ 5x\times 3\times 5=75x \\ 4a\times 3\times b=12ab \end{array}\right\}\text{new numerators;}\qquad \left\{\begin{array}{l}\text{Hence the fractions required are} \\ \frac{10bx}{15b},\ \frac{75x}{15b},\ \frac{12ab}{15b}. \end{array}\right.$$

$$3\times b\times 5=15b\ \text{common denominator;}$$

Ex. 2.

Reduce $\frac{2x+1}{5}$, and $\frac{3x}{4}$, to a common denominator.

$$\begin{array}{l} \text{Here } (2x+1)\times 4=8x+4 \\ \qquad 3x\times 5=15x \end{array}\ \text{new numerators;}\qquad \text{Hence the fractions required are}$$

$$5\times 4=20\ \text{common denominator;}\qquad \frac{8x+4}{20},\ \&\ \frac{15x}{20}.$$

Ex. 3.

Reduce $\frac{5x}{a+x}$, $\frac{a-x}{3}$, and $\frac{1}{2x}$, to a common denominator.

$$\begin{array}{l} \text{Here } 5x\times 3\times 2x=30x^2 \\ (a-x)\times(a+x)\times 2x=2a^2x-2x^3 \\ 1\times(a+x)\times 3=3a+3x \\ (a+x)\times 3\times 2x=6ax+6x^2 \end{array}\qquad \begin{array}{l} \therefore \text{the new fractions are} \\ \frac{30x^3}{6ax+6x^2},\ \frac{2a^2x-2x^3}{6ax+6x^2}, \\ \text{and } \frac{3a+3x}{6ax+6x^2}. \end{array}$$

Ex. 4.

Reduce $\frac{3x}{5}$, $\frac{4bx}{3a}$, and $\frac{5x^2}{a}$, to a common denominator.

Answer, $\frac{9a^2x}{15a^2}$, $\frac{20abx}{15a^2}$, and $\frac{75ax^2}{15a^2}$.

Ex. 5.

Reduce $\frac{2x+3}{x}$, and $\frac{5x+1}{3}$, to a common denominator.

Answ. $\frac{6x+9}{3x}$, and $\frac{5x^2+x}{3x}$.

Ex. 6.

Reduce $\frac{4x^2+2x}{5}$, $\frac{3x^2}{4a}$, and $\frac{2x}{3b}$, to a common denominator.

Answ. $\frac{48abx^2+24abx}{60ab}$, $\frac{45bx^2}{60ab}$, and $\frac{40ax}{60ab}$.

Ex. 7.

Reduce $\frac{7x^2-1}{2x}$, and $\frac{4x^2-x+2}{2a^2}$, to a common denominator.

Answ. $\frac{14a^2x^2-2a^2}{4a^2x}$, and $\frac{8x^3-2x^2+4x}{4a^2x}$.

35. *To reduce a Fraction to its lowest terms.*

Rule. "Observe what quantity will divide all the terms "both of the numerator and denominator *without a remainder;* Divide them by this quantity, and the fraction is reduced to its lowest terms." A more general Rule will be given at the end of this Chapter.

Example 1.

Reduce $\frac{14x^3+7ax+21x^2}{35x^2}$ to its lowest terms.

The coefficient of every term of the numerator and denominator of this fraction is divisible by 7, and the letter x also enters into every term; therefore $7x$ will divide both numerator and denominator without a remainder.

Now $\frac{14x^3+7ax+21x^2}{7x}=2x^2+a+3x$,

and $\frac{35x^3}{7x}=5x^2$;

hence, the fraction in its lowest terms is $\frac{2x^2+a+3x}{5x}$.

Ex. 2.

Reduce $\frac{20abc-5a^2+10ac}{5a^2c}$ to its lowest terms.

Here the quantity which divides both numerator and denominator without a remainder is $5a$; the fraction therefore in its lowest terms is $\frac{4bc-a+2c}{ac}$.

Ex. 3. Reduce $\frac{a-b}{a^2-b^2}$ to its lowest terms.

Here $a-b$ will divide both numerator and denominator, for by Ex. 2. Case III. page 13. $a^2-b^2=(a+b)(a-b)$; hence $\frac{1}{a+b}$ is the fraction in its lowest terms.

Ex. 4. Reduce $\frac{10x^3}{15x^2}$ to its lowest terms.

ANSWER, $\frac{2x}{3}$.

Ex. 5. . . . . . $\frac{3abx^2}{6ax}$ to its lowest terms.

ANSW. $\frac{bx}{2}$.

Ex. 6. . . . . . $\frac{14x^2y^2-21x^3y^2}{7x^3y}$ to its lowest terms.

ANSW. $\frac{2y-3xy}{x}$.

Ex. 7. . . $\frac{51x^3-17x^2+34x}{17x^6}$ to its lowest terms.

ANSW. $\frac{3x^2-x+2}{x^4}$.

Ex. 8. Reduce $\frac{a-b}{a^3-b^3}$ to its lowest terms. (See Art. 31.)

Answ. $\frac{1}{a^2+ab+b^2}$.

X.

ON THE ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF FRACTIONS.

36. *To add Fractions together.*

Rule. "Reduce the fractions to a common denominator, "and then add their numerators together; bring the resulting fraction to its lowest terms, and it will be the "sum required."

Example 1.

Add $\frac{3x}{5}$, $\frac{2x}{7}$, and $\frac{x}{3}$, together.

$$\left.\begin{array}{r} 3x \times 7 \times 3 = 63x \\ 2x \times 5 \times 3 = 30x \\ x \times 5 \times 7 = 35x \\ \hline 5 \times 7 \times 3 = 105 \end{array}\right| \therefore \frac{63x + 30x + 35x}{105} = \frac{128x}{105}$$

is the fraction required.

Ex. 2. Add $\frac{a}{b}$, $\frac{2a}{3b}$, and $\frac{5b}{4a}$, together.

$$\left.\begin{array}{r} a \times 3b \times 4a = 12a^2b \\ 2a \times b \times 4a = 8a^2b \\ 5b \times b \times 3b = 15b^3 \\ \hline b \times 3b \times 4a = 12ab \end{array}\right\} \begin{array}{l} \therefore \frac{12a^2b + 8a^2b + 15b^3}{12ab} = \frac{20a^2b + 15b^3}{12ab^2} \\ = \text{(dividing by } b\text{)} \frac{20a^2 + 15b^2}{12ab} \end{array}$$

is the sum required.

Ex. 3. Add $\frac{2x+3}{5}$, $\frac{3x-1}{2x}$, and $\frac{4x}{7}$, together.

$$\left.\begin{array}{r} (2x+3) \times 2x \times 7 = 28x^2 + 42x \\ (3x-1) \times 5 \times 7 = 105x - 35 \\ 4x \times 5 \times 2x = 40x^2 \\ \hline 5 \times 2x + 7 = 70x \end{array}\right\} \begin{array}{l} \therefore \frac{28x^2 + 42x + 105x - 35 + 40x^2}{70x} \\ = \frac{68x^2 + 147x - 35}{70x} \end{array}$$

is the sum required.

Ex. 4. Add $\frac{3x}{7}$, $\frac{5x}{9}$, and $\frac{4x}{11}$, together.

ANSWER, $\frac{934x}{693}$.

Ex. 5. . . . $\frac{3a^2}{2b}$, $\frac{2a}{5}$, and $\frac{3b}{7a}$, together.

ANSW. $\frac{105a^3+28a^2b+30b^2}{70ab}$.

Ex. 6. . . . $\frac{2x+1}{3}$, $\frac{4x+2}{5}$, and $\frac{x}{7}$, together.

ANSW. $\frac{169x+77}{105}$.

Ex. 7. . . . $\frac{5a^2+b}{3b}$, and $\frac{4a^2+2b}{5b}$, together.

ANSW. $\frac{37a^2+11b}{15b}$.

Ex. 8. . . . $\frac{2x-5}{3}$, and $\frac{x-1}{2x}$, together.

ANSW. $\frac{4x^2-7x-3}{6x}$.

Ex. 9. . . . $\frac{x}{x-3}$, and $\frac{x}{x+3}$, together.

ANSW. $\frac{2x^2}{x^2-9}$.

Ex. 10. . . . $\frac{a+b}{a-b}$, and $\frac{a-b}{a+b}$, together.

ANSW. $\frac{2a^2+2b^2}{a^2-b^2}$.

37. *To Subtract Fractional Quantities.*

RULE. "Reduce the fractions to a common denominator; and then subtract the numerators from each other "and under the difference write the common denominator."

EXAMPLE 1.

Subtract $\frac{3x}{5}$ from $\frac{14x}{15}$.

$$\left.\begin{array}{r} 3x\times 15=45x \\ 14x\times\ 5=70x \\ \hline 5\ \times 15=75 \end{array}\right\} \therefore \frac{70x-45x}{75}=\frac{25x}{75}=\frac{x}{3} \text{ is the difference required.}$$

Ex. 2

Ex. 2.

Subtract $\frac{2x+1}{3}$ from $\frac{5x+2}{7}$

$$\left.\begin{array}{l}(2x+1)\times 7=14x+7\\(5x+2)\times 3=15x+6\\\hline \quad 3\ \times 7=21\end{array}\right\}\therefore \frac{15x+6-14x-7}{21}=\frac{x-1}{21}$$ is the fraction required.

Ex. 3.

From $\frac{10x-9}{8}$ subtract $\frac{3x-5}{7}$.

$$\left.\begin{array}{l}(10x-9)\times 7=70x-63\\(\ 3x-5)\times 8=24x-40\\\hline \quad 8\ \times 7=56\end{array}\right\}\therefore \frac{70x-63-24x+40}{56}=\frac{46x-23}{56}$$ is the fraction required.

Ex. 4.

From $\frac{a+b}{a-b}$ subtract $\frac{a-b}{a+b}$.

$$\left.\begin{array}{l}(a+b)(a+b)=a^2+2ab+b^2\\(a-b)(a-b)=a^2-2ab+b^2\\\hline (a-b)(a+b)=a^2-b^2\end{array}\right\}\therefore \frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}$$ is the fraction required.

Ex. 5. Subtract $\frac{4x}{5}$ from $\frac{9x}{10}$ . . . . Answer, $\frac{x}{10}$.

Ex. 6. . . . . . $\frac{5x+1}{7}$ from $\frac{21x+3}{4}$.

Answ. $\frac{127x+17}{28}$.

Ex. 7. . . . . . $\frac{3x+1}{x+1}$ from $\frac{4x}{5}$. Answ. $\frac{4x^2-11x-5}{5x+5}$.

Ex. 8. . . . . . $\frac{2x-3}{3x}$ from $\frac{4x+2}{3}$. Answ. $\frac{4x^2+3}{3x}$.

Ex. 9. . . . . . $\frac{1}{a+b}$ from $\frac{1}{a-b}$. Answ. $\frac{2b}{a^2-b^2}$.

Ex. 10. . . . . . $\frac{3x-7}{8}$ from $\frac{4x}{7}$. Answ. $\frac{11\text{[illegible]}+49}{56}$.

38. *To Multiply Fractional Quantities.*

Rule. " Multiply their numerators together for a new " numerator, and their denominators together for a new " denominator, and reduce the resulting fraction to its " lowest terms."

EXAMPLE 1.

Multiply $\frac{2x}{7}$ by $\frac{4x}{9}$.

$$\left.\begin{aligned}2x\times4x&=8x^2\\ 7\times9&=63\end{aligned}\right\}\ \therefore\ \text{the fraction required is } \frac{8x^2}{63}.$$

Ex. 2. Multiply $\frac{4x+1}{3}$ by $\frac{6x}{7}$.

Here

$(4x+1)\times6x=24x^2+6x$

and

$3\times7\ =21$

$\frac{24x^2+6x}{21}$: (dividing the numerator and denominator by 3)

$\frac{8x^2+2x}{7}$ is the fraction required.

Ex. 3. Multiply $\frac{a^2-b^2}{5b}$ by $\frac{3a^2}{a+b}$.

By Ex. 2. Case III. page 13. $(a^2-b^2)\times3a^2=(a+b)$ $(a-b)\times3a^2$; hence the product is $\frac{(3a^2\times(a+b)(a-b)}{5b\times(a+b)}=$ (dividing numerator and denominator by $a+b$) $\frac{3a^2\times(a-b)}{5b}$ $=\frac{3a^3-3a^2b}{5b}$.

Ex. 4. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$.

Here

$(3x^2-5x)\times7a=21ax^2-35ax$

and

$(2x^3-3x)\times14=28x^3-42x$

$\therefore\ \frac{21ax^2-35ax}{28x^3-42x}=$ (dividing the numerator and denominator by $7x$) $\frac{3ax-5a}{4x^2-6}$ is the fraction required.

Ex. 5. Multiply $\frac{2x}{x-1}$ by $\frac{3x}{7}$. . . Answer, $\frac{6x^2}{7x-7}$.

Ex. 6. . . . . . $\frac{3x^2-x}{5}$ by $\frac{10}{2x^2-4x}$. Answ. $\frac{3x-1}{x-2}$.

Ex. 7. . . . . . $\frac{2a}{a-b}$ by $\frac{a^2-b^2}{8}$. Answ. $\frac{a^2+ab}{4}$.

Ex. 8. . . . . . $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$. Answ. $\frac{9x}{2}$.

39. *On the Division of Fractions.*

Rule. "*Invert* the divisor, and proceed as in Multiplication."

Ex. 1. Divide $\frac{14x^2}{9}$ by $\frac{2x}{3}$.

Invert the divisor, and it becomes $\frac{3}{2x}$; hence $\frac{14x^2}{9}\times\frac{3}{2x}$ $=\frac{42x^2}{18x}=\frac{7x}{3}$ (dividing the numerator and denominator by $6x$) is the fraction required.

Ex. 2. Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$.

$$\frac{14x-3}{5}\times\frac{25}{10x-4}=\frac{(14x-3)\times 5}{10x-4}=\frac{70x-15}{10x-4}.$$

Ex. 3. Divide $\frac{5a^2-5b^2}{2a}$ by $\frac{4a+4b}{6b}$.

$$\frac{5a^2-5b^2}{2a}=\frac{5(a+b)(a-b)}{2a}$$

$$\frac{4a+4b}{6b}=\frac{4\times(a+b)}{6b};$$

$$\therefore\ \frac{5(a+b)\,(a-b)}{2a}\times\frac{6b}{4\times(a+b)}$$

$$=\frac{30b\,(a-b)}{8a}=\frac{15ab-15b^2}{4a}$$ is the fraction required.

Ex. 4. Divide $\frac{4x}{7}$ by $\frac{9x}{5}$ . . . . Answer, $\frac{20}{63}$.

Ex. 5. . . . . $\frac{4x+2}{3}$ by $\frac{2x+1}{5x}$. Answ. $\frac{10x}{3}$.

Ex. 6. Divide $\frac{x^2-9}{5}$ by $\frac{x+3}{4}$. Answer, $4x-12$

Ex. 7. $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. Answ. $\frac{9x-3}{}$.

XI.

On the Method of finding the Greatest Common Measure of two or more Quantities.

40. One quantity is said to *measure* another, when it is contained in that other a certain number of times, without a remainder.

41. A quantity is said to be a *multiple* of another, when it contains that other quantity a certain number of times, without a remainder.

42. A *common measure* of two or more quantities is any quantity which measures them all; and the *greatest* common measure is the greatest quantity which will so measure them. Thus, $2a$ is a common measure of the quantities $24ab^2$, $16a^2bc$, and $12abc^2$, and their *greatest* common measure is $4ab$.

43. If one quantity measures another, it will also measure any *multiple* of that quantity. Thus, let b measure a by the units in m, then $a=mb$, and let na be a multiple (denoted by the units in n) of a; the $na=nmb$; consequently b measures na by the units in nm.

44. If one quantity measures two others, it will also measure their sum and difference. For let c measure a by the units in m, and b by the units in n, then $a=mc$, and $b=nc$; therefore $a\pm b$[a] $=mc\pm nc, =(m\pm n)c$; consequently c measures $a+b$ (their *sum*) by the units in $m+n$, and $a-b$ (their *difference*) by the units in $m-n$.

45. The

(a) The quantity $a\pm b$ means *a plus* or *minus b*.

45. The Rule for finding the greatest common measure of two *numbers* may be thus investigated. Let a and b be any two numbers, whereof a is the greater; and let the following operation be performed upon them; viz.

$$\left.\begin{array}{l} b)a(p \\ \;\;\underline{pb} \\ \;\;c)b(q \\ \;\;\;\;\underline{qc} \\ \;\;\;\;d)c(r \\ \;\;\;\;\;\;\underline{rd} \\ \;\;\;\;\;\;\;0 \end{array}\right\}$$

Where a divided by b gives the *quotient* p, and *remainder* c; b divided by c, the quotient q, and remainder d; c divided by d, the quotient r, and remainder 0. Then, since in each case the *dividend* is equal to the *divisor* multiplied by the *quotient* plus the *remainder*, we have,

$c=rd$

$b=qc+d=$ (since $qc=qrd$) $qrd+d=(qr+1)d$

$a=pb+c=\left(\begin{array}{ll}\text{since } pb=(pqr+p)d \\ \text{and} \quad c= \quad rd\end{array}\right)(pqr+p+r)d.$ Hence since p, q, r are *whole numbers*, d is contained in b as many times as there are units in $qr+1$, and in a as many times as there are units in $pqr+p+r$; consequently the *last divisor* d is a common measure of a and b; and this is evidently the case, whatever be the length of the operation, provided that it be carried on till the remainder is nothing.

This last divisor d is also the *greatest* common measure of a and b. For let x be any common measure of a and b, such that

$a=mx$, and $b=nx$, then

$c=a-pb=mx-pnx=(m-pn)x$

$d=b-qc=nx-(qm-pqn)x=(n-qm+pqn)x$; $\therefore x$ measures d by the units in $n-qm+pqn$, that is, *every* common measure of a and b measures d. Now it has been shewn that d is a common measure of a and b; and the greatest measure of d is evidently *itself;* consequently d is the greatest common measure of a and b. Hence this Rule for finding the greatest common measure of two numbers; "Divide the greater by the lesser, and the preceding "divisor by the last remainder, till nothing remains; the "last divisor is the greatest common measure."

To find the greatest common measure of *three* numbers, a, b, c; let d be the greatest common measure of a and b, and x the greatest common measure of d and c; then x is the greatest common measure of a, b, and c. For, let $a=md$, $b=nd$, $d=px$; then $a=mpx$, and $b=npx$, therefore x is a common measure of a and b; and, since it also measures c, it will be a common measure of a, b, and c. But, as above, *every* common measure of a and b measures d; therefore *every* common measure of a, b, and c, measures d and c; and, consequently, the greatest common measure of d and c, or x, will also be the greatest common measure of a, b, and c.

In general, let there be any set of numbers, a, b, c, d, e, &c.; and let x be the greatest common measure of a and b; y the greatest common measure of x and c; z the greatest common measure of y and d; &c. &c.; then will y be the greatest common measure of a, b, c,; z the greatest common measure of a, b, c, d; &c. &c.

46. To find the greatest *simple* common measure of *Algebraic* quantities, the Rule is, "to find the greatest "common measure of their coefficients, and then annex "to it the letters common to all the quantities;" thus the greatest common measure of $24\,ax^2y^2$, $16bxy$, and $6axy^2$ is $2xy$.

To find the greatest *compound* common measure of two algebraic quantities, "first divide each of them by their "greatest *simple* common measure (if they have one); "arrange their terms according to the dimensions of the "same letter, and divide either, or both of them, by the "greatest simple factor which it may contain; then perform on them the same operation as that for finding "the greatest common measure of two *numbers*, observing "only, that the remainders which arise are to be divided "by their greatest simple factors, and that the dividends "may, if requisite, be multiplied by any simple quantity "which will make the first term of the dividend a multiple "of the first term of the divisor. Lastly, multiply the "compound common measure thus obtained by the *simple* "one originally taken out, and the product will be the "greatest common measure required."(a)

Example 1.

Find the greatest common measure of $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$.

These quantities having no simple divisors, we immediately proceed as follows;

$$\begin{array}{r} 6a^2+7ax-3x^2\big)\,6a^2+11ax+3x^2\,\big(1 \\ \underline{6a^2+7ax-3x^2} \\ +4ax+6x^2 \end{array}$$

Dividing $4ax+6x^2$ by its greatest simple divisor $2x$, we have

$$\begin{array}{l} 2a+3x\big)\,6a^2+7ax-3x^2\,\big(3a-x \\ \underline{6a^2+9ax} \\ -2ax-3x^2 \\ \underline{-2ax-3x^2} \\ *\quad * \end{array}$$

Hence $2a+3x$ is the greatest common measure.

Ex. 2.

Find the greatest common measure of $8a^2b^2-10ab^3+2b^4$ and $9a^4b-9a^3b^2+3a^2b^3-3ab^4$.

The greatest simple common measure of these quantities is b; which being taken out from both, they become

---

(a) The rejection of these simple factors from the original quantities, and from the remainders which arise in the process, or the multiplication of the dividends pointed out in the Rule, will not affect the compound common measure sought; which can have no simple factor, because the original quantities have (by the Rule) their simple factors taken out, previously to this part of the process.

$8a^2b-10ab^2+2b^3$ and $9a^4-9a^3b+3a^2b^2-3ab^3$; the former of these is divisible by $2b$, and the latter by $3a$; which divisions being made, the given quantities are reduced to $4a^2-5ab+b^2$, and $3a^3-3a^2b+ab^2-b^3$. Multiply this last by 4, to make the operation succeed, and we have

$$\begin{array}{r}
4a^2-5ab+b^2\big)\,12a^3-12a^2b+4ab^2-4b^3\,\big(3a \\
\underline{12a^3-15ab+3ab^2} \\
3ab+\ ab^2-4b^3
\end{array}$$

Dividing the remainder by b, and multiplying the new dividend by 3, we have

$$\begin{array}{r}
3a^2+ab-4b^2\big)\,12a^2-15ab+\ 3b^2\,\big(4 \\
\underline{12a^2+\ 4ab-16b^2} \\
-19ab+19b^2
\end{array}$$

Lastly, Divide the remainder by $-19b$, and proceed thus;

$$\begin{array}{r}
a-b\big)\,3a^2+ab-4b^2\,\big(3a-4b \\
\underline{3a^2-3ab} \\
4ab-4b^2 \\
\underline{4ab-4b^2} \\
*\quad *
\end{array}$$

which gives $a-b$ for the *compound* common measure; and this being multiplied into the *simple* one b, we have $ab-b^2$ for the greatest common measure sought.

CHAP. III.

ON THE

INVOLUTION AND EVOLUTION OF NUMBERS

AND OF

ALGEBRAIC QUANTITIES.

XII.

On the Involution of Numbers and Simple Algebraic Quantities.

47. *Involution*, or "the raising of a quantity to a given power," is performed by the continued multiplication of that quantity into itself, till the number of factors amounts to the number of units in the index of that given power. Thus, the *square* of a or $a^2 = a \times a$; the *cube* of b or $b^3 = b \times b \times b$; the *fourth power* of $2 = 2 \times 2 \times 2 \times 2 = 16$; the *fifth power* of $3 = 3 \times 3 \times 3 \times 3 \times 3 = 243$; &c. &c. This rule, as applied to *numbers*, will be readily understood by the mere inspection of the following Table.

ROOTS AND POWERS OF NUMBERS.

Root	1	2	3	4	5	6	7	8	9	10
Square	1	4	9	16	25	36	49	64	81	100
Cube	1	8	27	64	125	216	343	512	729	1000
4th power	1	16	81	256	625	1296	2401	4096	6561	10000
5th power	1	32	243	1024	3125	7776	16807	32768	59049	100000

48. The operation is performed in the same manner for simple algebraic quantities; except that in this case it must be observed, that the powers of *negative* quantities are alternately + and −; the *even* powers being positive, and the *odd* powers *negative*. Thus the *square* of $+2a$

is $+2a \times +2a$ or $+4a^2$; the square of $-2a$ is $-2a \times -2a$ or $+4a^2$; but the *cube* of $-2a = -2a \times -2a \times -2a = +4a^2 \times -2a = -8a^3$.

The several powers of $\frac{a}{b}$ are,	And the several powers of $-\frac{b}{2c}$,
Square $=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$,	*Square* $=-\frac{b}{2c}\times-\frac{b}{2c}=+\frac{b^2}{4c^2}$,
Cube $=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\frac{a^3}{b^3}$,	*Cube* $=-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}=-\frac{b^3}{8c^3}$,
4th power $=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^4}{b^4}$,	*4th power* $=-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}=+\frac{b^4}{16c^4}$.
&c. =&c.	

Upon this principle the powers of the several roots in the following Table are calculated.

ROOTS AND POWERS OF SIMPLE ALGEBRAIC QUANTITIES.

Root	a	$-b$	$2b^2$	$\frac{a}{2b}$	$-\frac{3x^2}{y}$	$\frac{2a}{3b}$	a^2b	$-\frac{a}{b}$	$-\frac{3x}{5}$	$\frac{x}{4y}$
Square	a^2	$+b^2$	$4b^4$	$\frac{a}{4b}$	$+\frac{9x^4}{y^2}$	$\frac{4a^2}{9b^2}$	a^4b^2	$+\frac{a^4}{b^6}$	$+\frac{9x^2}{25}$	$\frac{x^2}{16y}$
Cube	a^3	$-b^3$	$8b^6$	$\frac{a^3}{8b^3}$	$-\frac{27x^6}{y^3}$	$\frac{8a^3}{27b^3}$	a^6b^3	$-\frac{a^6}{b^9}$	$-\frac{27x^3}{125}$	$\frac{x}{64y^3}$
4th Power	a^4	$+b^4$	$16b^8$	$\frac{a^4}{16b^4}$	$+\frac{81x^8}{y^4}$	$\frac{16a^4}{81b^4}$	a^8b^4	$+\frac{a^8}{b^{12}}$	$+\frac{81x^4}{625}$	$\frac{x^4}{256y^4}$
5th Power	a^5	$-b^5$	$32b^{10}$	$\frac{a^5}{32b^5}$	$-\frac{243x^{10}}{y^5}$	$\frac{32a^5}{243b^5}$	$a^{10}b^5$	$-\frac{a^{10}}{b^{15}}$	$-\frac{243x^5}{3125}$	$\frac{x^5}{1024y^5}$

XIII.

On the Involution of Compound Algebraic Quantities.

49. The powers of compound algebraic quantities are raised by the mere application of the Rule for Compound Multiplication (Art. 22.) Thus,

Ex. 1. What is the square of $a+2b$?

$$\begin{array}{rl}
 & a+2b \\
 & a+2b \\ \hline
 & a^2+2ab \\
 & \quad\;+2ab+4b^2 \\ \hline
\text{Square}= & \underline{\underline{a^2+4ab+4b^2}}
\end{array}$$

Ex. 2. What is the cube of a^2-x?

$$\begin{array}{rl}
 & a^2-x \\
 & a^2-x \\ \hline
 & a^4-a^2x \\
 & \quad\;-a^2x+x^2 \\ \hline
\text{Square}= & a^4-2a^2x+x^2 \\
 & a^2-x \\ \hline
 & a^6-2a^4x+a^2x^2 \\
 & \quad\;-a^4x+2a^2x^2-x^3 \\ \hline
\text{Cube}= & \underline{\underline{a^6-3a^4x+3a^2x^2-x^3}}
\end{array}$$

Ex. 3

What is the 5th power of $a+b$?

$$\begin{array}{l}
a+b \\
a+b \\ \hline
a^2+ab \\
\quad\;+ab+b^2 \\ \hline
a^2+2ab+b^2=\text{Square} \\
a+b \\ \hline
a^3+2a^2b+ab^2 \\
\quad\;+a^2b+2ab^2+b^3 \\ \hline
a^3+3a^2b+3ab^2+b^3=\text{Cube} \\
a+b \\ \hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\quad\;+a^3b+3a^2b^2+3ab^3+b^4 \\ \hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4=\text{4th Power} \\
a+b \\
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\quad\;+a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\ \hline
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=\text{5th Power.}
\end{array}$$

Ex. 4. The 4th *power* of $a+3b$ is $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$.

Ex. 5. The *square* of $3x^2+2x+5$ is $9x^4+12x^3+34x^2+20x+25$.

Ex. 6. The *cube* of $3x-5$ is $27x^3-135x^2+225x-125$.

Ex. 7. The *cube* of x^2-2x+1 is $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

50. In the involution of a binomial quantity of the form $a+b$, the several terms in each successive power are found to bear a certain relation to each other, and observe a certain law, which the following Table is intended to explain.

TABLE OF THE POWERS OF $a+b$.

Powers	Mode of expressing them.	Powers expanded.
Square	$(a+b)^2$	$a^2+2ab+b^2$.
Cube	$(a+b)^3$	$a^3+3a^2b+3ab^2+b^3$.
4th Power	$(a+b)^4$	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
5th Power	$(a+b)^5$	$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.
6th Power	$(a+b)^6$	$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

The successive powers of $a-b$ are precisely the same as those of $a+b$, except that the signs of the terms will be alternately + and −. Thus, the fourth power of $a-b$ is $a^4-4a^3b+6a^2b^2-4ab^3+b^4$; and so of the rest.

In reviewing that column of the foregoing Table, which contains the powers of $a+b$ *expanded*, we may observe,

I. That in each case, the *first* term is a raised to the *given power*, and the *last* term is b raised to the *same* power; thus, in the *square*, the *first* term is a^2, and the *last* b^2; in the cube, the *first* term is a^3, and the *last* b^3; and so of the rest.

II. That, with respect to the intermediate terms, the powers of a *decrease*, and the powers of b *increase*, by unity in each successive term. Thus, in the fifth power, we have

In the *second* term . . . . a^4b;
third . . . . a^3b^2;
fourth . . . . a^2b^3;
fifth . . . . ab^4;

and so in the *other* powers.

III. That in each case, the *coefficient of the second term* is the same with the *index of the given power*. Thus, in the square it is 2; in the cube it is 3; in the fourth power it is 4; and so of the rest.

IV. That if the *coefficient* of a in any term be multiplied by its *index*, and the product divided by the *number of terms to that place*, the quotient will give the *coefficient* of the *next term*. Thus,

In the *fourth* power, $\frac{\text{coeff. of } a \text{ in the } 2^d \textit{ term} \times \textit{its index}}{\text{number of terms to that place}}$

$=\frac{4\times 3}{2}=\frac{12}{2}=6=$coefficient of *third term*.

In the *sixth* power, $\frac{\text{coeff. of } a \text{ in the } 4^{th} \textit{ term} \times \textit{its index}}{\text{number of terms to that place}}$

$=\frac{20\times 3}{4}=\frac{60}{4}=15=$coefficient of *fifth term*.

We are thus furnished with a general Rule for raising the binomial $a+b$ to any power, without the process of actual multiplication. For instance, let it be required to raise $a+b$ to the *eighth* power; then, according to the Rule just laid down,

The *first* term is . . . . a^8.
The *second* . . . . $8a^7b$.
The *third* . . . . $\frac{8\times 7}{2}a^6b^2=28a^6b^2$.
The *fourth* . . . . $\frac{28\times 6}{3}a^5b^3=56a^5b^3$.
The *fifth* . . . . $\frac{56\times 5}{4}a^4b^4=70a^4b^4$;

and so on.

And thus we have
$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$.

In the same manner it will be found,

Ex. 2. That $(a-b)^7 = a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$.

Ex. 3. That $(x-y)^9 = x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$.

Ex. 4. That $(x+a)^{10} = x^{10} + 10x^9a + 45x^8a^2 + 120x^7a^3 + 210x^6a^4 + 252x^5a^5 + 210x^4a^6 + 120x^3a^7 + 45x^2a^8 + 10xa^9 + a^{10}$.

In reviewing these several examples, it may be observed, that, when the number of terms in the resulting quantity is *even*, the coefficients of the two middle terms are the *same ;* and that *in all cases*, the coefficients *increase* as far as the *middle* terms, and then *decrease* precisely in the same manner until we come to the last term. By attending to this *law of the coefficients*, it will only be necessary to calculate them as far as the *middle term*, and then set down the rest in an *inverted order*. Thus, in Ex. 3. $(\overline{x-y}^{\,9})$.

The *first* five coefficients are 1, 9, 36, 84, 126.
The *last* five . . . . 126, 84, 36, 9, 1.

51. But we are not yet arrived at the *most general* form in which this Rule may be exhibited. Suppose it was required to raise the binomial $a+b$ to any power denoted by the number (n). Proceeding with n as we have done with the several indices in the preceding examples, it appears that

The *first* term would be a^n.
The *second* . . . . $na^{n-1}b$.
The *third* . . . . $\frac{n(n-1)}{2}a^{n-2}b^2$.
The *fourth* . . . . $\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3$.
The *fifth* . . . . $\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}b^4$.

The *last* term would be b^n.

Or, $(a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{2.3} a^{n-3} b^3 + \frac{n(n-1)(n-2)(n-3)}{2.3.4} a^{n-4} b^4 + \&c. \;\ldots\ldots\; + b^n$.

By the same process, $(a-b)^n = a^n - n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 - \frac{n(n-1)(n-2)}{2.3} a^{n-3} b^3 + \&c.$; the signs of the terms being alternately + and −.

This general and compendious method of raising a binomial quantity to any given power, is called, from the name of its celebrated inventor, Sir I. NEWTON's "Binomial Theorem." Its use will appear from the following Examples.

EXAMPLE 1.

Raise $x^2 + 3y^2$ to the *fifth* power.

In comparing $(x^2+3y^2)^5$ with $(a+b)^n$, we have, $a=x^2$, $b=3y^2$, $n=5$.

Substituting these quantities for a, b, n, in the foregoing general formula, it appears, that

The *first* term . . (a^n) . . . . is $(x^2)^5$ . . . . $= x^{10}$.

2d . . . . . . $(n a^{n-1} b)$ . . . . is $5 \times (x^2)^4 \times 3y^2 = 15 x^8 y^2$.

3d . . . . . . $\left(\frac{n(n-1)}{2} a^{n-2} b^2\right)$ . . . . is $5 \times \frac{4}{2} \times (x^2)^3 \times (3y^2)^2 = 90 x^6 y^4$.

4th . . . . $\left(\frac{n(n-1)(n-2)}{2.3} a^{n-3} b^3\right)$ is $5 \times \frac{4}{2} \times \frac{3}{3} \times (x^2)^2 \times (3y^2)^3 = 270 x^4 y^6$.

5th, $\left(\frac{n(n-1)(n-2)(n-3)}{2.3.4} a^{n-4} b^4\right)$ is $5 \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times x^2 \times (3y^2)^4 = 405 x^2 y^8$.

Last . . . . . (b^n) . . . . is $(3y^2)^5 = 243 y^{10}$.

So that $(x^2+3y^2)^5 = x^{10} + 15x^8y^2 + 90x^6y^4 + 270x^4y^6 + 405x^2y^8 + 243y^{10}$.

In the application of this formula, it may be observed, that the *number of terms*, of which the binomial consists, is always *one more* than the *index of the given power;* after having calculated therefore as many terms as there are units in the index of the given power, we may immediately proceed to the *last term*.

Ex. 2.

Raise $3x+2y$ to the 6th power.

Here $\left.\begin{array}{r}3x=a\\ 2y=b\\ n=6\end{array}\right\}$ and $(3x+2y)^6=729x^6+2916x^5y+4860x^4y^2+4320x^3y^3+2160x^2y^4+576xy^5+64y^6$.

Ex. 3.

Raise $x-2y^2$ to the 7th power.

Here $\left.\begin{array}{r}x=a\\ 2y^2=b\\ n=7\end{array}\right\}$ and comparing $(x-2y^2)^7$ with $(a-b)^n$, we have $x^7-14x^6y^2+84x^5y^4-280x^4y^6+560x^3y^8-672x^2y^{10}+448xy^{12}-128y^{14}$ for the quantity required.

52. By means of this Theorem, we are enabled to raise a *trinomial* or *quadrinomial* quantity to any power, without the process of actual multiplication. Thus, suppose it was required to *square* $a+b+c$; inclosing $a+b$ in a parenthesis $(a+b)$, and considering it as *one* quantity, we should have $(a+b+c)^2=\overline{(a+b)+c}|^2=(a+b)^2+2(a+b)c+c^2$ $=a^2+2ab+b^2+2ac+2bc+c^2$.

In the same manner we have,

Ex. 1. $(a+b+c+d)^2=\overline{(a+b)+(c+d)}|^2=(a+b)^2+2(a+b)$ $(c+d)+(c+d)^2=a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2$ $+2cd+d^2=a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$.

Ex. 2. $(a+b+c)^3=\overline{(a+b)+c}|^3=(a+b)^3+3(a+b)^2c+$ $3(a+b)c^2+c^3=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+$ $3ac^2+3bc^2+c^3=a^3+b^3+c^3+3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)$ $+6abc$.

Ex. 3. $(x+y+3z)^2=\overline{(x+y)+3z}|^2=(x+y)^2+2(x+y)\times$ $3z+(3z)^2=x^2+2xy+y^2+6xz+6yz+9z^2$.

XIV.

On the Evolution of Algebraic Quantities.

53. *Evolution,* "or the Rule for extracting the root of any quantity," is just the reverse of *Involution.* To perform the operation, we must inquire what quantity multiplied into

itself, till the number of factors amount to the number of units in the index of the given root, will generate the quantity whose root is to be extracted.

54. This Rule, as applied to small numbers and *simple* algebraic quantities, may be easily explained by reference to the Tables in Art^s. 47, 48. Thus,

$49=7\times 7$; $\therefore$ the *square root* of 49, or $\sqrt{49}=7$.

$-b^3=-b\times -b\times -b$; $\therefore$ the *cube root* of $-b^3=(\sqrt[3]{-b^3})=-b$.

$\frac{16a^4}{81b^4}=\frac{2a}{3b}\times\frac{2a}{3b}\times\frac{2a}{3b}\times\frac{2a}{3b}$; $\therefore$ the *4th* or *biquadrate* root of $\frac{16a^4}{81b^4}\left(\sqrt[4]{\frac{16a^4}{81b^4}}\right)=\frac{2a}{3b}$

$32=2\times 2\times 2\times 2\times 2$; $\therefore$ the *fifth* root of 32 $(\sqrt[5]{32})=2$.

&c. &c.

55. If the quantity unde[illegible]he radical sign does not admit of resolution into the number of factors indicated by that sign, or, in other words, if it be not a *complete power*, then its exact root cannot be extracted, and the quantity itself, with the radical Sign annexed, is called a *Surd*. Thus, $\sqrt{37}$, $\sqrt[3]{a^2}$, $\sqrt[4]{b^3}$, $\sqrt[5]{47}$, &c. &c. are Surd quantities. The application of the fundamental rules of arithmetic to quantities of this kind will form the subject of Chap. VIII.

56. In the involution of *negative* quantities, it was observed that the *even* powers were all +, and the *odd* powers −; there is consequently no quantity which, multiplied into itself in such manner that the number of factors shall be *even*, can generate a negative quantity. Hence quantities of the form $\sqrt{-a^2}$, $\sqrt[4]{-10}$, $\sqrt[6]{-a^3}$, $\sqrt{-5}$, $\sqrt[4]{-a^4}$, &c. &c. have no real root, and are therefore called *impossible*.

57. In extracting the roots of *compound* quantities, we must observe in what manner the terms of the *root* may be derived from those of the power. For instance (by Art. 50), the square of $a+b$ is $a^2+2ab+b^2$, where the terms are arranged according to the powers of a. On comparing $a+b$ with $a^2+2ab+b^2$, we observe that the first term of the power (a^2) is the square of the first term of the root (a).

Put a therefore for the first term of the root, square it, and subtract that square from the first term of the power. Bring down the other two terms $2ab+b^2$, and *double* the first term of the root; set down $2a$, and having divided the first term of the remainder $(2ab)$ by it, it gives b, the other term of the root; and since $2ab+b^2=(2a+b)b$, if to $2a$ the term b is added, and this sum multiplied by b, the result is $2ab+b^2$; which being subtracted from the two terms brought down, nothing remains.

$$\begin{array}{r|l} & a^2+2ab+b^2(a+b \\ & a^2 \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline & * \quad\; * \end{array}$$

58. Again, the square of $a+b+c$ (Art. 52.) is $a^2+2ab+b^2+2ac+2bc+c^2$; in this case, the root may be derived from the power, by continuing the process in the last Article. Thus having found the two first terms $(a+b)$ of the root as before, we bring down the remaining three terms $2ac+2bc+c^2$ of the power; and dividing $2ac$ by $2a$, it gives c, the third term of the root. Next, let the last term (b) of the preceding divisor be doubled, and add c to the divisor thus increased, and it becomes $2a+2b+c$; multiply this new divisor by c, and it gives $2ac+2bc+c^2$, which being subtracted from the three terms last brought down, leaves no remainder. In this manner the following Examples are solved.

$$\begin{array}{r|l} & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c \\ & a^2 \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline 2a+2b+c & 2ac+2bc+c^2 \\ & 2ac+2bc+c^2 \\ \hline & * \quad\; * \quad\; * \end{array}$$

Ex. 1. $4x^4+6x^3+\frac{89}{4}x^2+15x+25\left(2x^2+\frac{3}{2}x+5.\right.$

$$\begin{array}{r|l} & 4x^4 \\ \hline 4x^2+\frac{3}{2}x & 6x^3+\frac{89}{4}x^2 \\ & 6x^3+\frac{9}{4}x^2 \\ \hline 4x^2+3x+5 & 20x^2+15x+25 \\ & 20x^2+15x+25 \end{array}$$

Ex. 2. $x^6+4x^5+2x^4+9x^2-4x+4(x^3+2x^2-x+2$

$$\begin{array}{l}
2x^3+2x^2)\,4x^5+2x^4 \\
\qquad\qquad\quad 4x^5+4x^4 \\
2x^3+4x^2-x)-2x^4+9x^2-4x \\
\qquad\qquad\qquad\quad -2x^4-4x^3+x^2 \\
2x^3+4x^2-2x+2)+4x^3+8x^2-4x+4 \\
\qquad\qquad\qquad\qquad\quad +4x^3+8x^2-4x+4
\end{array}$$

59. The process for extracting the *Cube* Root of a compound quantity may be explained in the following manner. By Art. 50, the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the terms being arranged according to the powers of a. The first term of the root is a, which being cubed, and this cube subtracted from the first term in the power (a^3), bring down the remaining three terms $3a^2b+3ab^2+b^3$. Next square the first term (a) of the root, and having multiplied it by 3, place $3a^2$ in the divisor, divide $3a^2b$ by $3a^2$, and it gives b the second term of the root; to $3a^2$ add $3ab+b^2$, and it forms the divisor $3a^2+3ab+b^2$, which being multiplied by b gives $3a^2b+3ab^2+b^3$; subtract, and nothing remains.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3\,(a+b \\
 & a^3 \\
\hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
 & * \quad * \quad *
\end{array}$$

60. The cube root of a compound quantity, if that root consists of three terms, is found by continuing the process in a similar manner.

$$\begin{array}{r|l}
 & (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3\,(a+b+c \\
 & (a+b)^3 \\
\hline
3(a+b)^2+3(a+b)c+c^2 & 3(a+b)^2c+3(a+b)c^2+c^3 \\
 & 3(a+b)^2c+3(a+b)c^2+c^3
\end{array}$$

Thus (by Art. 52) the cube of $a+b+c$ is $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$; supposing the first two terms of the root to have been found as in the preceding article, cube $a+b$ and subtract $(a+b)^3$ from the first term of the power; and then bring down the next three terms $3(a+b)^2c+3(a+b)c^2+c^3$. Square the two terms already found; which square being multiplied by 3, gives $3(a+b)^2$; divide $3(a+b)^2c$ by $3(a+b)^2$, and we have c, the third term of the root. To $3(a+b)^2$ add $3(a+b)c+c^2$, and it forms the divisor $3(a+b)^2+3(a+b)c+c^2$, which being multiplied by c, gives $3(a+b)^2c+3(a+b)c^2+c^3$; subtract, and nothing remains.

If the quantity whose root is required be not an exact power, the operation will not terminate as in the above instances; but it may be continued to any number of terms at pleasure.

Ex. Find the square root of a^2+x^2.

$$\begin{array}{ll}
 & a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^6}, \text{ \&c.}\right. \\
 & a^2 \\
\hline
2a+\frac{x^2}{2a}\Big) & x^2 \\
 & x^2+\frac{x^4}{4a^2} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big) & -\frac{x^4}{4a^2} \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\quad\Big) & \frac{x^6}{8a^4}-\frac{x^8}{64a^6} \text{ \&c.}
\end{array}$$

In these cases, however, the root is in general much more easily found by help of the Binomial Theorem, as will be explained hereafter.

XV.

On the Investigation of the Rules for the Extraction of the Square and Cube Roots of Numbers.

Before we proceed to the investigation of these Rules, it will be necessary to explain the nature of the common arithmetical notation.

61. It is very well known, that the value of the figures in the common arithmetical scale increases in a tenfold proportion from the right to the left; a number, therefore, may be expressed by the addition of the *units*, *tens*, *hundreds*, &c. of which it consists. Thus the number 4371 may be expressed in the following manner, viz. $4000+300+70+1$, or by $4\times1000+3\times100+7\times10+1$; hence, if the digits(a) of a number be represented by a, b, c, d, e, &c. beginning from the left hand, then,

A number of 2 figures may be expressed by $10a+b$.
. . . . 3 figures . . . . by $100a+10b+c$.
. . . . 4 figures . . . . by $1000a+100b+10c+d$.
&c. &c. &c.

62. Let a number of three figures (viz. $100a+10b+c$) be squared, and its root extracted according to the Rule in Art. 58, and the operation will stand thus;

$$\begin{array}{l}
\text{I.}\quad 10000a^2+2000ab+100b^2+200ac+20bc+c^2(100a+10b+c\\
\quad 10000a^2\\
\overline{200a+10b)2000ab+100b^2}\\
\qquad\qquad 2000ab+100b^2\\
\qquad\qquad \overline{200a+20b+c)200ac+20bc+c^2}\\
\qquad\qquad\qquad\qquad\qquad 200ac+20bc+c^2
\end{array}$$

Let $\left.\begin{array}{l}a=2\\ b=3\\ c=1\end{array}\right\}$ and the operation is transformed into the following one;

$$\begin{array}{l}
40000+12000+900+400+60+1(200+30+1\\
\underline{40000}\\
\overline{400+30)12000+900+400}\\
\qquad\qquad \underline{12000+900}\\
\qquad\qquad 400+60+1)400+60+1\\
\qquad\qquad\qquad\qquad\qquad 400+60+1
\end{array}$$

(a) By the *digits* of a number are meant the figures which compose it, considered independently of the value which they possess in the arithmetical scale.

III. But it is evident that this operation would not be affected by collecting the several numbers which stand in the same line into one sum, and leaving out the ciphers which are to be *subtracted* in the several parts of the operation. Let this be done; and let two figures be brought down at a time, after the square of the first figure in the root has been subtracted; then the operation may be exhibited in the manner annexed; from which it appears that the square root of 53361 is 231.

```
      ·  · ·
      53361 (231
      4
   43|133
     |129
  461|461
     |461
```

63. To explain the division of the given number into *periods* consisting of two figures each, by placing a dot over every second figure, beginning with the units (as exhibited in the foregoing operation), it must be observed, that, since the square root of 100 is 10; of 10000 is 100; of 1000000 is 1000; &c. &c. it follows, that the square root of a number *less than* 100 must consist of *one* figure; of a number *between* 100 *and* 10000, of *two* figures; of a number between 10000 and 1000000, of *three* figures; &c. &c. and consequently the number of these dots will shew the number of figures contained in the square root of the given number. From hence it also follows, that the *first* figure of the root will be the square root of the greatest square number contained in the first of those periods, reckoning from the *left*. Thus, in the case of $\dot{5}3\dot{3}6\dot{1}$ (whose square root is a number consisting of *three* figures); since the square of the figure standing in the *hundred's* place cannot be found either in the *last* period (61), or in the *last but one* (33), it must be found in the *first* period (5); consequently the first figure of the root will be the square root of the *greatest* square number contained in 5; and as this number is 4, the first figure of the root will be 2.

---

scale. Thus the digits of the number 537 are simply the numbers 5, 3, and 7; whereas the 5, considered with respect to its place in the numeration scale, means 500, and the 3 means 30.

The remainder of the operation will be readily understood, by comparing the steps of it with the several steps of the process for finding the square root of $(a+b+c)^2$ in Art. 58; for having subtracted 4 for the first period (5), there remains 1; bring down the next two figures (33), and the dividend is 133; double the first figure of the root (2), and place the result 4 in the divisor; 4 is contained in 13 three times, 3 is therefore the second figure of the root; place this both in the divisor and quotient, and the former is 43; multiply by 3, and subtract 129, the remainder is 4; to which bring down the next two figures (61), which gives 461 for the next dividend. Lastly, double the last figure of the former divisor, and it becomes 46; place this in the next divisor, and since 4 is contained in 4 *once*, 1 is the third figure of the root; place 1 therefore both in the divisor and quotient; multiply and subtract as before, and nothing remains.

64. The rule for extracting the *cube* root of numbers may be understood by comparing the process for extracting the cube root of $(a+b+c)^3$ in Art. 59. and 60, with the following operations, in which is deduced the cube root of the number 13997521.

$$\begin{array}{lr}
 & 1\dot{3}99\dot{7}52\dot{1}\ \big(200+40+1 \\
 & a^3=(200)^3=8000000 \\
\hline
3a^2=120000) & \text{1st Remainder } 5997521 \\
\hline
 & 3a^2b=3\times(200)^2\times40=4800000 \\
 & 3ab^2=3\times200\times(40)^2=\ \ 960000 \\
 & b^3=40\times40\times40\ =\ \ \ 64000 \\
\hline
 & 5824000 \\
\hline
3(a+b)^2=172800) & \text{2d Remainder } \ 173521 \\
\hline
 & 3(a+b)^2c=3(200+40)^2\times1=172800 \\
 & 3(a+b)c^2=3(200+40)\ \times1=\ \ \ \ 720 \\
 & c^3=\ \ \ 1\times1\times1=\ \ \ \ \ 1 \\
\hline
 & 173521 \\
 & \text{3d Remainder } 000000
\end{array}$$

Omitting the superfluous ciphers, and bringing down three figures at a time, the operation would stand thus;

$$\begin{array}{rr}
 & 1\dot{3}99\dot{7}52\dot{1}\,(241 \\
2^3= & 8 \\
\hline
3\times 2^2=12) & 5997 \\
\hline
300\times 2^2\times 4= & 4800 \\
30\times 2\times 4^2= & 960 \\
4^3= & 64 \\
\hline
 & 5824 \\
\hline
3\times 24^2=1728) & 173521 \\
\hline
300\times(24)^2\times 1= & 172800 \\
30\times 24\times 1= & 720 \\
1^3= & 1 \\
\hline
 & 173521 \\
\hline
 & 000000 \\
\hline
\end{array}$$

These operations may be explained in the following manner.

I. Since the cube root of 1000 is 10, of 1000000 is 100, &c. it follows, that the cube root of a number less than 1000 will consist of *one* figure; of a number between 1000 and 1000000 of *two* figures, &c. &c.; if therefore the given number be divided into *periods*, each consisting of *three figures*, by placing a dot over every third figure beginning with the units, the number of those dots will shew the number of figures of which the cube root consists; and for the reason assigned in the preceding article (respecting the first figure of the square root), the *first figure* of the root will be the cube root of the greatest cube number contained in the first period.

II. Having *pointed* the number, we find that its cube root consists of *three* figures. The *first* figure is the cube root of the greatest cube number contained in 13; this being 2, the value of this figure is 200, or $a=200$; consequently $a^3=8000000$;

subtract this number from 13997521, and the remainder is 5997521. Find the value of $3a^2$, and divide this latter number by it, and it gives 40 for the value of b the *second* member of the root; put this in the quotient, and then calculate the value of $3a^2b+3ab^2+b^3$ and subtract it, and there remains 173521. Find now the value of $3(a+b)^2$, and divide 173521 by it, and it gives 1 for the value of c the *third* member of the root; put this in the quotient, and then calculate the amount of $3(a+b)^2c+3(a+b)c^2+c^3$, which subtract, and nothing remains.

III. In reviewing the *first* of these two operations, it is evident that *six* ciphers might have been rejected in the value of a^3, and *three* in the value of $3a^2b+3ab^2+b^3$, without affecting the substance of the operation; having therefore simplified the process as in the *second* operation, we are furnished with the following Rule for extracting the cube root of numbers.

IV. "Point off every *third* figure, beginning with the "units; find the greatest cube number contained in the "*first* period, and place the cube root of it in the quotient; "*cube* it and *subtract* it from the first period, and then "bring down the next three figures; divide the number "thus brought down by 300 times the square of the first "figure of the root, and it will give the *second* figure; "then calculate the value of 300 × square of first figure "× second figure + 30 × first figure × square of second "+ cube of second, subtract it, and then bring down the "next period, and so proceed till all the periods are "brought down." The Rules for extracting the higher powers of numbers and of compound algebraic quantities are very tedious, and of no great practical utility.

XVI.

On the General Mode of expressing the Powers and Roots of Quantities by Means of Indices.

65. The management of *Surd* quantities, and the method of extracting the roots of compound algebraic quantities by means of the Binomial Theorem, will be treated of hereafter;

but before we conclude this chapter, it may be proper to make a few observations on the method of expressing the powers and roots of quantities by means of *indices*.

I. Since $a \times a^2 = a^3 = a^{1+2}$; $a^2 \times a^3 = a^5 = a^{2+3}$; or, in general, $a^m \times a^n = a^{m+n}$, it follows, that the different powers of any quantity are *multiplied* together by *adding the indices*.

II. Again, $\frac{a^2}{a} = a = a^{2-1}$; $\frac{a^5}{a^3} = a^2 = a^{5-3}$; or, in general, $\frac{a^m}{a^n} = a^{m-n}$; from which it appears, that one power of a is *divided* by another, by *subtracting* the index of the *divisor* from that of the *dividend*.

III. The Square of $a = a \times a = a^{1\times2} = a^2$,

Cube of $a^2 = a^2 \times a^2 \times a^2 = a^{2\times3} = a^6$,

or, in general, mth power of $a^n = a^n \times a^n \times a^n$ to m factors $= a^{mn}$; from this it follows, that the powers of a are *raised* to other powers by *multiplying* the index of the original power by that of the power to which it is to be raised.

IV. Square root of $a^2 = a^1 = a^{\frac{2}{2}}$

Square root of $a^4 = a^2 = a^{\frac{4}{2}}$,

Cube root of $a^6 = a^2 = a^{\frac{6}{3}}$, &c. &c., i.e. the *roots* of powers of a are found by *dividing* the *index* of the *power* by the number expressing the degree of the root to be taken.

66. From this method of considering the formation of the powers and roots of quantities, a new species of algebraic notation arises, of which the following are examples.

I. The *roots* or quantities may be expressed by *fractional indices*. Thus,

The Square root of $a = a^{1\div2} = a^{\frac{1}{2}}$;

Cube root of $a = a^{1\div3} = a^{\frac{1}{3}}$;

or, in general, mth root of $a = a^{1\div m} = a^{\frac{1}{m}}$.

Again, Cube root of $a^2 = a^{2\div3} = a^{\frac{2}{3}}$;

Square root of $a^3 = a^{3\div2} = a^{\frac{3}{2}}$;

5th root of $a^2 = a^{2\div5} = a^{\frac{2}{5}}$;

or, in general, mth root of $a^n = a^{n\div m} = a^{\frac{n}{m}}$.

II. The signification of the *negative indices* arising from Rule 4 of Division (Art. 23) will easily appear by an example. By that Rule, $\frac{a^2}{a^5}=a^{2-5}=a^{-3}$. But $\frac{a^2}{a^5}=\frac{1}{a^3}$; consequently a^{-3} and $\frac{1}{a^3}$ (and, in general, a^{-m} and $\frac{1}{a^m}$) are equivalent expressions. Hence it follows that a^0 will always represent unity, whatever be the value of a; for, by the Rule, $\frac{a^m}{a^m}=a^{m-m}$, or $1=a^0$.

A comparison of the following series, in the first of which every succeeding term is the quotient of the preceding divided by a, and, in the second, the index of a is continually diminished by 1, will shew that the above conclusions naturally follow from the notation adopted in Art. 7.

$$\begin{array}{ccccccc} aaa & aa & a & 1 & \frac{1}{a} & \frac{1}{aa} & \frac{1}{aaa} \\ a^3 & a^2 & a^1 & a^0 & a^{-1} & a^{-2} & a^{-3} \end{array}$$

III. From this it follows, that any factor may be removed from the *numerator* of a fraction into the *denominator*, or from the *denominator* into the *numerator*, by *changing the sign* of its index.

Ex. 1. Thus (since $\frac{1}{b^3}=b^{-3}$) $\frac{a^2}{b^3}$ may be expressed by a^2b^{-3}; and $\left(\text{since } a^2=\frac{1}{a^{-2}}\right)$, we have $\frac{a^2}{b^3}=\frac{1}{a^{-2}}\times\frac{1}{b^3}=\frac{1}{a^{-2}b^3}$.

Ex. 2. The quantity $\frac{a^{\frac{1}{2}}b}{c^{\frac{2}{3}}d^4e^{\frac{4}{5}}}$ may be expressed by $a^{\frac{1}{2}}b^3c^{-\frac{2}{3}}d^{-4}e^{-\frac{4}{5}}$, or by $\frac{1}{a^{-\frac{1}{2}}b^{-3}c^{\frac{2}{3}}d^4e^{\frac{4}{5}}}$.

CHAP. IV.

ON SIMPLE EQUATIONS.

When two algebraic quantities are connected together by the sign of equality, the whole expression thus formed is called (Art. 11.) an *Equation*. Equations, as applied to the solution of questions or problems, consist of quantities, some of which are *known*, and others *unknown*; and by the *solution* of an equation is meant, the operation by which the value of the unknown quantities are found in terms of the known ones. If any equation contains no *power* of the unknown quantities, but those quantities merely in their simplest form, it is called a *Simple* Equation; if it contains the *square* of the unknown quantity, it is called a *Quadratic* Equation; if the *cube* of the unknown quantity, a *Cubic* Equation; &c. &c. The present chapter will be occupied entirely with the solution of *Simple* Equations, and questions depending upon them.

XVII.

On the Solution of Simple Equations, containing only one unknown quantity.

67. The Rules absolutely necessary for the solution of simple equations containing only one unknown quantity may be reduced to four, and may be arranged in the following order.

Rule I.

The first Rule is, that "any quantity may be transferred "from one side of the equation to the other, by changing its sign; and it is founded upon the axiom, that "if equals "be *added* to or *subtracted* from equals, the *sums* or *remainders* will be equal."

Ex. 1. Let $x+8=15$; *subtract* 8 from each side of the equation, and it becomes $x+8-8=15-8$; but $8-8=0$, $\therefore x=15-8=7$.

Ex. 2. Let $x-7=20$; *add* 7 to each side of the equation, then $x-7+7=20+7$; but $-7+7=0$, $\therefore x=20+7=27$.

Ex. 3. Let $3x-5=2x+9$; *add* 5 to each side of the equation, and it becomes $3x-5+5=2x+9+5$, or $3x=2x+9+5$. *Subtract* $2x$ from each side of this latter equation, then $3x-2x=2x-2x+9+5$; but $2x-2x=0$, $\therefore 3x-2x=9+5$. Now $3x-2x=x$, and $9+5=14$; hence $x=14$.

On reviewing the steps of these examples, it appears

I. That $x+8=15$ is equivalent to . . . . $x=15-8$.
II. . . . $x-7=20$ . . . . to . . . . $x=20+7$.
III. . . $3x-5=2x+9$ . . . . . . to $3x-2x=9+5$.

Or, that "the equality of the quantities on each side of the "equation is not affected by removing a quantity from one "side of the equation to the other, and *changing its sign*."

From this Rule also it appears, that if the same quantity with the same sign be found on *both* sides of an equation, it may be left out of the equation; thus, if $x+a=c+a$, then $x=c+a-a$; but $a-a=0$, $\therefore x=c$.

It further appears, that the signs of *all* the terms of an equation may be changed from + to −, or from − to +, without altering the value of the unknown quantity. For let $x-b=c-a$; then, by the Rule, $x=c-a+b$; change the signs of *all* the terms, then $b-x=a-c$, in which case $b-a+c=x$, or $x=c-a+b$, as before.

RULE

RULE II.

"If the unknown quantity has a coefficient, then its "value may be found by dividing each side of the equa- "tion by that coefficient;" and the foundation of the Rule is, "that if equals be divided by the same, the quotients "arising will be equal."

Ex. 1. Let $2x=14$: then *di*[illegible]both sides of the equation by 2, we have $\frac{2x}{2}=\frac{14}{2}$; but $\frac{2x}{2}=x$, and $\frac{14}{2}=7$, $\therefore x=7$.

Ex. 2. Let $6x+10=3x+22$; then, by RULE I, $6x-3x=22-10$, or $3x=12$; *divide* each side by 3, then $\frac{3x}{3}=\frac{12}{3}$, or $x=4$.

Ex. 3. Let $ax=b+c$; then $\frac{ax}{a}=\frac{b+c}{a}$; but $\frac{ax}{a}=x$, $\therefore x=\frac{b+c}{a}$.

RULE III.

"An equation may be cleared of fractions by multi- "plying each side of the equation by the denominators of "the fractions in succession, or by their product." This Rule goes upon the principle, that "if equals be multi- "plied by the same, the products arising will be equal."

Ex. 1. Let $\frac{x}{3}=6$; *multiply* each side of the equation by 3, then (since, from what has been already shewn, the multiplication of the fraction $\frac{x}{3}$ by 3, just takes away its denominator, and gives x) we have $x=6\times 3=18$.

Ex. 2. Let $\frac{x}{2}+\frac{x}{5}=7$; *multiply* each side of the equation by 2, and we have $x+\frac{2x}{5}=14$; now multiply each side by 5, and it becomes $5x+2x=70$, or $7x=70$; hence, by RULE II, $x=\frac{70}{7}=10$.

Ex. 3. Let $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$.

Multiply each side of equation by 2, then $x+\frac{2x}{3}=26-\frac{2x}{4}$

. . . . by 3, and $3x+2x=78-\frac{6x}{4}$.

. . . . by 4, . . $12x+8x=312-6x$.

By Rule I, $12x+8x+6x=312$

or $26x=312$

$\therefore$ by Rule II, $x=\frac{312}{26}=12$.

This Example might have been solved more simply, by multiplying each side of the equation by the product of the numbers 2, 3, 4, which is 24.

Thus, $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$.

Multiply each side by 24, then $\frac{24x}{2}+\frac{24x}{3}=312-\frac{24x}{4}$,

or $12x+8x=312-6x$, as before.

Rule IV.

"If the equation contains the square root of the unknown "quantity, or the square root of the unknown quantity "combined with some known quantity; then, let this surd "quantity be brought by itself to one side of the equation, "and let both sides of the equation be *squared;* the value "of the unknown quantity may then be found by the pre-"ceding Rules." This Rule goes upon the supposition, that "if the *square root* of a quantity be equal to any "given quantity, then the *quantity itself* will be equal to "the *square* of that given quantity."

Ex. 1. Let $\sqrt{x}-5=3$; then by Rule I, $\sqrt{x}=5+3=8$; *square* both sides of the equation, then $x=8\times 8=64$.

Ex. 2. Let $\sqrt{2x+1}+2=5$; then, by Rule I, $\sqrt{2x+1}=5-2=3$; *square* both sides of the equation, and we have $2x+1=9$, $\therefore 2x=9-1=8$, and $x=\frac{8}{2}=4$.

68. The following Examples will serve to exercise the learner in these several Rules.

In Rule I.

Ex. 1. $2x+3 = x+17$ . . . . . Answer, $x=14$.
Ex. 2. $5x-4 = 4x+25$ . . . . $x=29$.
Ex. 3. $7x-9 = 6x-3$ . . . . $x=6$.
Ex. 4. $4x+2a=3x+9b$ . . . . $x=9b-2a$.

In Rules I, II.

Ex. 1. $10x=150$ . . . . . Answer, $x=15$.
Ex. 2. $15x+4=34$ . . . . $x=2$.
Ex. 3. $8x+7=6x+27$ . . . . $x=10$.
Ex. 4. $9x-3=4x+22$ . . . . $x=5$.
Ex. 5. $17x-4x+9=3x+39$ . . . . $x=3$.
Ex. 6. $ax-c=b+2c$ . . . . $x=\frac{b+3c}{a}$

In Rules I, II, III.

Ex. 1. $\frac{2x}{3}+\frac{x}{4}=22$ . . . . Answer, $x=24$.
Ex. 2. $\frac{7x}{4}-\frac{5x}{6}=\frac{55}{6}$ . . . . $x=10$
Ex. 3. $\frac{x}{2}+\frac{x}{3}=31-\frac{x}{5}$ . . . . $x=30$
Ex. 4. $\frac{2x}{5}-\frac{x}{6}+\frac{x}{2}=44$ . . . . $x=60$.

In Rule IV.

Ex. 1. $\sqrt{x}-1=4$ . . . . Answer, $x=25$.
Ex. 2. $\sqrt{3x+1}+5=10$ . . . . $x=8$.
Ex. 3. $15+\sqrt{x+7}=19$ . . . . $x=9$.

69. In the application of these Rules to the solution of simple equations in general containing only one unknown quantity, it will be proper to observe the following method.

I. To clear the equation of fractions by Rule III.

II. To collect the *unknown* quantities on one side of the equation, and the *known* on the other, by RULE I.

III. To find the value of the unknown quantity by dividing each side of the equation by its coefficient, as in RULE II.

IV. If the equation contains a *surd* quantity, then RULE IV. must be immediately applied.

EXAMPLE 1.

Find the value of x in the equation $\frac{3x}{7}+1=\frac{x}{5}+\frac{13}{5}$.

Multiply by 7, then $3x+7=\frac{7x}{5}+\frac{91}{5}$;

. . . . . by 5, . . $15x+35=7x+91$.

Collect the *unknown* quantities on *one* side, and the known on the *other*; $\Big\}$ $15x-7x=91-35$, or $8x=56$.

Divide by the coefficient of x, $x=\frac{56}{8}=7$.

Ex. 2.

Find the value of x in the equation $\frac{x+3}{5}-1=2-\frac{x}{7}$.

Multiply by 5, then $x+3-5=10-\frac{5x}{7}$;

. . . . . by 7, . . $7x+21-35=70-5x$.

Collect the *unknown* quantities on *one* side, and the known on the *other*; $\Big\}$ $7x+5x=70-21+35$, or $12x=84$;

$\ldots\frac{84}{12}=7$.

Ex. 3.

Find the value of x in the equation

$$4x-\frac{x-1}{\text{[illegible]}}=x+\frac{2x-2}{\text{[illegible]}}+24.$$

Multiply by the *product* (10), $\Big\}$ $40x-5x+5=10x+4x-4+240$(a).

By transposition, $40x-5x-10x-4x=240-4-5$,

$$\text{or } 40x-19x=231,$$

$$\text{i. e. } 21x=231\,;\ \therefore\ x=\frac{231}{21}=11.$$

Ex. 4.

Find the value of x in the equation $2x-\frac{x}{2}+1=5x-2$.

Multiply by 2, then $4x-x+2=10x-4$.

By transposition, $4+2=10x-4x+x$,

$$\text{or } 6=7x\,;\ \therefore\ x=\frac{6}{7}.$$

Ex. 5.

What is the value of x in the equation $3ax+2bx=3c+a$?

$$\text{Here } 3ax+2bx=(3a+2b)\times x\,;$$

$$\therefore\ (3a+2b)\times x=3c+a.$$

Divide each side of the equation by $3a+2b$, which is the coefficient of x; then $x=\frac{3c+a}{3a+2b}$.

Ex. 6.

Find the value of x in the equation $3bx+a=2ax+4c$. Bring the *unknown* quantities to *one* side of the equation, and the *known* to the *other*; then,

$$3bx-2ax=4c-a$$

$$\text{but } 3bx-2ax=(3b-2a)\times x\,;$$

$$\therefore\ (3b-2a)x=4c-a.$$

Divide by $3b-2a$, and $x=\frac{4c-a}{3b-2a}$.

Ex. 7.

(a) As this step involves the case "where the sign — stands before a Fraction," when the numerator of that fraction is brought down into the same line with $40x$, the signs of both its terms must be *changed*, for the reasons assigned in Ex. 3, page 26; and we therefore make it $-5x+5$, and not $-5x-5$.

Ex. 7.

Find the value of x in the equation $bx+x=2x+3a$.

Transpose $2x$, then $bx+x-2x=3a$,

or $bx-x=3a$,

but $bx-x=(b-1)x$;

$\therefore (b-1)x=3a$, or $x=\dfrac{3a}{b-1}$.

Ex. 8.

Find the value of x in the equation $\dfrac{3x}{a}-c+\dfrac{x}{b}=4x+\dfrac{2x}{d}$.

Multiply by abd, then $3bdx-abcd+adx=4abdx+2abx$.

By transposition, $3bdx+adx-4abdx-2abx=abcd$,

or $(3bd+ad-4abd-2ab)x=abcd$

$$\therefore x=\frac{abcd}{3bd+ad-4abd-2ab}.$$

Ex. 9.

Let $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}$ to find the value of x.

Multiply by $\sqrt{a+x}$, then $\sqrt{x}\times\sqrt{a+x}+a+x=2a$.

By *transposition*, $\sqrt{x}\times\sqrt{a+x}=2a-a-x=a-x$.

Square both sides, $x\times(a+x)=a^2-2ax+x^2$;

or $ax+x^2=a^2-2ax+x^2$;

$\therefore 3ax=a^2$

and $x=\dfrac{a^2}{3a}=\dfrac{a}{3}$.

Ex. 10.

Let $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$ to find the value of x.

Square both sides, and we have $a^2+2ax+x^2=a^2+x\sqrt{b^2+x^2}$,

or $2ax+x^2=x\sqrt{b^2+x^2}$.

Divide by x, $2a+x=\sqrt{b^2+x^2}$.

Square *again*, $4a^2+4ax+x^2=b^2+x^2$;

$\therefore 4a^2+4ax=b^2$,

or $4ax=b^2-4a^2$.

Hence, $x=\dfrac{b^2-4a^2}{4a}=\dfrac{b^2}{4a}-a$.

Ex. 11. $x+\frac{x}{2}+\frac{x}{3}=11$ . . . . . ANSWER, $x=6$.

Ex. 12. $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}=\frac{x}{2}+17$ . . . . . . $x=60$.

Ex. 13. $4x-20=\frac{3x}{7}+\frac{110}{7}$ . . . . . . $x=10$.

Ex. 14. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}$ . . . . $x=\frac{6}{7}$.

Ex. 15. $3x+\frac{1}{9}=\frac{x+3}{3}$ . . . . $x=\frac{1}{3}$.

Ex. 16. $\frac{3x}{7}-5=29-2x$ . . . . $x=14$.

Ex. 17. $6x-\frac{3x}{4}-9=5x$ . . . . $x=36$.

Ex. 18. $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$ . . . $x=12$

Ex. 19. $\frac{x-2}{2}+\frac{x}{3}=20-\frac{x-6}{2}$ . . . . . $x=18$.

Ex. 20. $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7$. $x=8$.

Ex. 21. $2ax+b=3cx+4a$ . . . . . . $x=\frac{4a-b}{2a-3c}$.

Ex. 22 $5ax-2b+4bx=2x+5c$ . . $x=\frac{5c+2b}{5a+4b-2}$.

Ex. 23. $bx+2x-a=3x+2c$ . . . . $x=\frac{2c+a}{b-1}$.

Ex. 24. $3x-a+cx=\frac{a+x}{3}-\frac{b-x}{a}$ . . $x=\frac{4a^2-3b}{8a+3ac-3}$.

XVIII.

XVIII.

On the Solution of Simple Equations containing two or more unknown Quantities.

For the solution of equations containing two or more unknown quantities, as many independent equations are required as there are unknown quantities. The two equations nece[illegible] the solution of the case, when *two* unknown quant[illegible] concerned, may be expressed in the following manner,

$$ax+by=c$$
$$a'x+b'y=c'$$

where a, b, c, a', b', c' represent *known* quantities, and x, y the *unknown* quantities whose values are to be found in terms of these known quantities.

70. There are three different Rules by which the value of one of these unknown quantities may be determined;

RULE I.

$$\left.\begin{array}{l}\text{Let } ax+by=c\ (A)\\ \text{and } a'x+b'y=c'(B)\end{array}\right\}\begin{array}{c}\text{be the two equations}\\ \text{to be solved.}\end{array}$$

Multiply equation (A) by a', then $aa'x+a'by=a'c\,(C)$
. . . . (B) by a, . . . $aa'x+ab'y=ac'\,(D)$

Subtract equation (D) from (C), then $(a'b-ab')\,y=a'c-ac'$

$$\therefore\ y=\frac{a'c-ac'}{a'b-ab'}.$$

From which we deduce the following Rule. "Multiply "the first equation by the coefficient of x in the second "equation, and the second equation by the coefficient of "x in the first equation; subtract the *last* of these result-"ing equations from the *first*, and there will arise an "equation which contains only y and known quantities, "from which the value of y is determined."

RULE II.

From equation (A), $ax=c-by$, $\therefore\ x=\dfrac{c-by}{a}$.

(B), $a'x=c'-b'y$, $\therefore\ x=\dfrac{c'-b'y}{a'}$

Putting these two values of x equal to each other, we have

$$\frac{c'-b'y}{a'}=\frac{c-by}{a}\text{; and } \therefore\ ac'-ab'y=a'c-a'by;$$

By transposition, $(a'b-ab')y=a'c-ac'$

$$\text{and } y=\frac{a'c-ac'}{a'b-ab'}.$$

From which it appears, that "if the value of x in the "*first* equation be put equal to its value in the *second*, "there will arise a new equation involving only y, from "which the same value of y is found as before.

Rule III.

From equation (A), $x=\frac{c-by}{a}$; substitute this value of x in equation (B), then $a'\times\frac{c-by}{a}+b'y=c'$

$$\text{or } a'c-a'by+ab'y=ac'$$

$$\therefore\ a'c-ac'=(a'b-ab')y$$

$$\text{and } y=\frac{a'c-a'c}{a'b-a'b}.$$

From which we infer, that "if the value of x found from "the *first* equation be substituted for it in the *second*, "there will arise an equation which gives the same value "of y as in the two former instances."

71. Having determined the value of y, the value of x may be found in each case, by substituting this value for y either in the first or second equation. The value of x in the first equation is $\frac{c-by}{a}$; but $y=\frac{a'c-ac'}{a'b-ab'}$, $\therefore\ x=\frac{c}{a}-\frac{b(a'c-ac')}{a(a'b-ab')}$ =(by reducing these fractions to a common denominator) $\frac{bc'-b'c}{a'b-ab'}$. The value of x in the *second* equation is $\frac{c'-b'y}{a'}$ $=\frac{c'}{a'}-\frac{b(a'c-ac')}{a'(a'b-ab')}$=(by reducing these fractions to a common denominator) $\frac{bc'-b'c}{a'b-ab'}$, as before.

72. From hence it appears, that in finding the value of y, *either* of the three Rules may be applied; and that in finding the value of x, the value of y so found may be substituted either in the *first* or *second* equation. In the choice of the Rule which may be most adapted to practical application, experience only can be our guide. It may further be observed, that there are cases in which RULE I. may be somewhat varied: for instance, if the given equations be,

$$ax+by=c \quad (A)$$
$$a'x-b'y=c' \quad (B)$$

Multiply equation (A) by b', then, $ab'x+bb'y=b'c$ (C)

. . . . (B) by b, . . . $(a'bx-bb'y=bc'$ (D)

Add equations (D) and (C), then $(ab'+a'b)\,x=b'c+bc'$

$$\text{and } x=\frac{b'c+bc'}{ab'+a'b}.$$

Having the value of x, the value of y may be found by one of the preceding methods.

73 The following examples are intended to illustrate each Rule separately.

EXAMPLE 1.

$$\text{Let } \left.\begin{array}{r} 5x+4y=55\ (A) \\ 3x-2y=31\ (B) \end{array}\right\} \text{to find the values of } x \text{ and } y.$$

BY RULE I,

Multiply (A) by 3, then $15x+12y=165$

. . . (B) by 5 . . $15x+10y=155$

. by *subtraction*, we have $2y=10$, or $y=\frac{10}{2}=5$.

Now, from equation (A) we have $x=\frac{55-4y}{5}=$(since $y=5$,

and ∴ $4y=20$) $\frac{55-20}{5}=\frac{35}{5}=7$.

Ex. 2.

$$\text{Let } \left.\begin{aligned} x+4y&=16\ (A) \\ 4x+y&=34\ (B) \end{aligned}\right\}$$

From equation (A), we have $x=16-4y$.

. . . . (B), ... $x=\dfrac{34-y}{4}$.

Hence, by Rule II. $\dfrac{34-y}{4}=16-4y$,

$$\text{or } 34-y=64-16y;$$

$$\therefore\ 15y=30 \text{ or } y=\frac{30}{15}=2.$$

It has already been shewn that $x=16-4y=$ (since $y=2$, and $\therefore 4y=8$) $16-8=8$.

Ex. 3.

$$\text{Let } \frac{x+2}{3}+8y=31\ (A)$$

$$\frac{y+5}{4}+10x=192\ (B)$$

Clear eq^{n}.(A) of fract^{s}. $x+2+24y=93$, or $x+24y=91\,(C)$

. . . . . . (B) . . . . $y+5+40x=768$, or $y+40x=763\,(D)$

From equation (C), $x=91-24y$; by Rule III, substitute this value of x in equation (D); then we have,

$$y+40(91-24y)=763$$

$$\text{or } y+3640-960y=763$$

$$\therefore 959y=3640-763=2877$$

$$\text{and } y=\frac{2877}{959}=3.$$

By referring to equation (C), we have $x=91-24y=$ (since $y=3$, and $\therefore 24y=72$) $91-72=19$.

Ex. 4.

$$\text{Let } 3x+4y=29\ (A).$$

$$17x-3y=36\ (B).$$

In this example, the Rule mentioned in Art. 72 may be applied.

Multiply equation (*A*) by 3, then $9x+12y=\ 87\ (C)$
. . . . (*B*) by 4 . . $\underline{68x-12y=144}\ (D)$

Add equation (*D*) to (*C*), then $77x=231$, or $x=\frac{231}{77}=3$.

From equation (*A*) we have $4y=29-3x=$ (since $x=3$, and $\therefore 3x=9$) $29-9=20$; hence $y=\frac{20}{4}=5$.

Ex. 5. $\left.\begin{aligned}4x+3y&=31\\3x+2y&=22\end{aligned}\right\}$ . . . . ANSWER, $\left\{\begin{aligned}x&=4\\y&=5.\end{aligned}\right.$

Ex. 6. $\left.\begin{aligned}3x+2y&=40\\2x+3y&=35\end{aligned}\right\}$ . . . . . . $\left\{\begin{aligned}x&=10\\y&=\ 5.\end{aligned}\right.$

Ex. 7. $\left.\begin{aligned}5x-4y&=19\\4x+2y&=36\end{aligned}\right\}$ . . . . $\left\{\begin{aligned}x&=7\\y&=4.\end{aligned}\right.$

Ex. 8. $\left.\begin{aligned}3x+7y&=79\\2y-\tfrac{1}{2}x&=\ 9\end{aligned}\right\}$ . . . . $\left\{\begin{aligned}x&=10\\y&=\ 7.\end{aligned}\right.$

Ex. 9. $\left.\begin{aligned}\frac{x+y}{3}+1&=\ 6\\\frac{x-y}{7}+3&=\ 4\end{aligned}\right\}$ . . . . $\left\{\begin{aligned}x&=11\\y&=\ 4\end{aligned}\right.$

Ex. 10. $\left.\begin{aligned}\frac{x+y}{3}-2y&=\ 2\\\frac{2x-4y}{5}+y&=\frac{23}{5}\end{aligned}\right\}$ . . . . $\left\{\begin{aligned}x&=11\\y&=\ 1.\end{aligned}\right.$

Ex. 11. $\left.\begin{aligned}\frac{2x-3}{2}+y&=\ 7\\5x-13y&=\frac{67}{2}\end{aligned}\right\}$ . . . . $\left\{\begin{aligned}x&=8\\y&=\tfrac{1}{2}.\end{aligned}\right.$

Ex. 12. $\left.\begin{aligned}\frac{3x-7y}{3}&=\frac{2x+y+1}{5}\\8-\frac{x-y}{5}&=6\end{aligned}\right\}$ . . . . . . $\left\{\begin{aligned}x&=13\\y&=\ 3.\end{aligned}\right.$

74. When *three* unknown quantities are concerned, the most general form under which simple equations can be expressed, is

$$ax+by+cz=d\ (E)$$
$$a'x+b'y+c'z=d'\ (F)$$
$$a''x+b''y+c''z=d''\ (G),$$

and the mode of solution may be conducted in the following manner.

I. Multiply eq^n. (E) by a', then $aa'x+a'by+a'cz=a'd\ (H)$

. . . . (F) by a . . . $aa'x+ab'y+ac'z=ad'\ (K)$

Subtract (K) from (H), then $(a'b-ab')y+(a'c-ac')z=a'd-ad$

By multiplying (F) by a'', and (G) by a', and subtracting the latter result from the former, we obtain in the same manner $(a''b'-a'b'')y+(a''c'-a'c'')z=a''d'-a'd''\ (M)$.

II. Next, let the coefficients of y and z, and the other known quantities in equation (L), be represented by α, β, γ respectively; and those in equation (M) by α', β', γ' respectively; then those equations may be reduced to the following form; viz.

$$\alpha y+\beta z=\gamma$$
$$\alpha' y+\beta' z=\gamma'.$$

From which, by making the proper substitutions in Rule I, and in Art. 71, we have

$$z=\frac{\alpha'\gamma-\alpha\gamma'}{\alpha'\beta-\alpha\beta'}.$$

$$y=\frac{\beta\gamma'-\beta'\gamma}{\alpha'\beta-\alpha\beta'}.$$

III. From equation (E), we have $x=\frac{d}{a}-\frac{by+cz}{a}$; in which substituting the values of y and z just now found, we obtain

$$x=\frac{d}{a}-\frac{b(\beta\gamma'-\beta'\gamma)+c(\alpha'\gamma-\alpha\gamma')}{a(\alpha'\beta-\alpha\beta')}.$$

This mode of operation might be easily extended to equations containing any number of unknown quantities.

EXAMPLE 1.

EXAMPLE 1.

$$\text{Let } \left.\begin{aligned} 2x+3y+4z&=29(E)\\ 3x+2y+5z&=32(F)\\ 4x+3y+2z&=25(G) \end{aligned}\right\}\begin{array}{l}\text{to find}\\ \text{the values}\\ \text{of } x, y, z.\end{array}$$

I. Multiply (E) by 3, then $6x+9y+12z=87(H)$

. . . . . (F) by 2 . . $6x+4y+10z=64(K)$.

Subtract (K) from (H) . . $5y+2z=23(L)$.

Multiply (F) by 4, then $12x+8y+20z=128$

. . . . . (G) by 3 . . $12x+9y+6z=75$

Subtract . . . . . $-y+14z=53(M)$.

II. Hence the given equations are reduced to,

$$\begin{aligned} 5y+2z&=23(L)\\ -y+14z&=53(M). \end{aligned}$$

Again . . . $5y+2z=23$

Multiply (M) by 5, then $-5y+70z=265$

By *addition* . . . . $72z=288$, or $z=\dfrac{288}{72}=4$.

From equation (M) . . . . . . $y=14z-53=56-53=3$.

III. From equation (E) . . $x=\dfrac{29-3y-4z}{2}=\dfrac{29-25}{2}=2$.

$$\text{Ex. 2.}\quad \left.\begin{aligned} x+y+z&=90\\ 2x+40&=3y+20\\ 2x+40&=4z+10 \end{aligned}\right\}\ \ldots\ldots\ \text{ANSWER}\ \left\{\begin{aligned} x&=35\\ y&=30\\ z&=25. \end{aligned}\right.$$

$$\text{Ex. 3.}\quad \left.\begin{aligned} x+y+z&=53\\ x+2y+3z&=105\\ x+3y+4z&=134 \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ \left\{\begin{aligned} x&=21\\ y&=6\\ z&=23. \end{aligned}\right.$$

XIX.

The Solution of Questions producing Simple Equations.

In the reduction and management of equations, we have proceeded by fixed and stated rules; but in the solution of *questions* we have no such rules to guide us. Every particular question requires a distinct process of reasoning, to bring it into an algebraic form; and nothing but practice and experience can produce expertness and facility in conducting this process. All that can be done for the learner in this case, is, to explain the manner in which the principles of this science may be made to bear upon questions in general; for as soon as they can be brought into the shape of *equations*, we have only to apply the foregoing Rules for finding the value of the unknown quantity or quantities. Before we proceed, therefore, to any actual examples, it may be proper to shew the relation which arithmetical and algebraic operations stand in to each other.

75. Suppose the following arithmetical question was proposed for solution; viz. "To divide the number 35 into "two such parts, that one part may exceed the other part "by 9." A person unacquainted with Algebra, might, with no great difficulty, solve this question in the following manner.

I. It appears, in the first place, that there must be a *greater* and a *lesser* part.

II. The greater part must exceed the lesser by 9.

III. But it is evident that the greater and lesser parts added together must be equal to the whole number 35.

IV. If then we substitute for the greater part its *equivalent*, viz. "*the lesser part increased by* 9," it follows, that the lesser part increased by 9, with the *addition* of the said lesser part, is equal to 35.

V. Or, in other words, that *twice* the lesser part, with the addition of 9, is equal to 35.

VI. Therefore, *twice the lesser part* must be equal to 35, *with* 9 *subtracted from* it.

VII. Hence, twice the lesser part is equal to 26.

VIII. From which we conclude, that the *lesser part* is equal to 26 *divided by* 2; i. e. to 13.

IX. And consequently, as the *greater* part exceeds the *lesser* by 9, it must be equal to 22.

But by adopting the method of algebraic notation, the different steps of this solution may be much more briefly expressed as follows.

I. Let the *lesser* part . . . . $=x$.
II. Then the *greater* part . . . . $=x+9$.
III. But the greater part + lesser part . . $=35$.
IV. $\therefore x+9+x$ . . . . $=35$.
V. or $2x+9$ . . . . $=35$.
VI. $\therefore 2x$ . . . . $=35-9$.
VII. or $2x$ . . . . $=26$.
VIII. $\therefore x$ (*lesser* part) . . . . $=\frac{26}{2}=13$.
IX. and $x+9$ (*greater* part) . . . . $=13+9=22$.

76. Having thus explained the manner in which the several steps in the solution of an arithmetical question may be expressed in the language of Algebra, we now proceed to its exemplification.

QUESTION I.

There are two numbers whose difference is 15, and their sum 59. What are the numbers?

As their *difference* is 15, it is evident that the greater number must exceed the lesser by 15.

Let, therefore, $x=$ the lesser number,
then will $x+15=$ the greater.
But their *sum* $=59$
$\therefore x+x+15=59$
or $2x+15=59$
and $2x=59-15=44$
$\therefore x=\frac{44}{2}=22$ the lesser number
and $x+15=22+15=37$ the greater.

QUESTION 2.

What two numbers are those whose difference is 9; and if three times the greater be added to five times the lesser, the sum shall be 35?

Let x=the *lesser* number;

then $x+9$=*greater* number.

And 3 *times* the greater$=3\times(x+9)=3x+27$.

5 *times* the lesser$=5x$.

But by the question, 3 times the greater + 5 times the lesser = 35.

Hence, $(3x+27)+(5x) \dots\dots\dots\dots =35$,

$\therefore\ 8x+27=35$,

or $8x=35-27=8$; $\therefore\ x=1$ *lesser* number,

and $x+9=1+9=10$ the *greater* number.

QUESTION 3.

What number is that to which 10 being added, $\frac{3}{5}$ths of the sum shall be 66?

Let x=the number required;

Then $x+10$=the number, with 10 added to it.

Now $\frac{3}{5}$ths of $(x+10)=\frac{3}{5}(x+10)=\frac{3(x+10)}{5}=\frac{3x+30}{5}$.

But, by the question, $\frac{3}{5}$ths of $(x+10)=66$;

Hence, $\frac{3x+30}{5}=66$.

Multiply by 5, then $3x+30=330$;

$\therefore\ 3x=330-30=300$; or $x=\frac{300}{3}=100$.

QUESTION 4.

What number is that which being multiplied by 6, the product increased by 18, and that sum divided by 9, the quotient shall be 20?

Let x= the number required;

then $6x$=the number multiplied by 6;

$6x+18$=the product increased by 18,

and $\frac{6x+18}{9}$=that sum divided by 9.

Hence, by the question, $\frac{6x+18}{9}=20$

Multiply by 9, then $6x+18=180$,

or $6x=180-18=162$; $\therefore x=\frac{162}{6}=27$.

QUESTION 5.

A post is $\frac{1}{5}$th in the earth, $\frac{3}{7}$ths in the water, and 13 feet out of the water. What is the length of the post?

Let x = the length of the post;

then $\frac{x}{5}$ = the part of it in the earth,

$\frac{3x}{7}$ = the part of it in the water,

13 = the part of it out of the water.

But part in earth + part in water + part out of water = whole post;

$$\left(\frac{x}{5}\right) \quad + \quad \left(\frac{3x}{7}\right) \quad + \quad 13=x.$$

Multiply by 5, then $x+\frac{15x}{7}+65 =5x$;

. . . . . by 7 . . . $7x+15x+455=35x$,

or $455=35x-7x-15x=13x$.

Hence $x=\frac{455}{13}=35$ = length of post.

QUESTION 6.

After paying away $\frac{1}{4}$th and $\frac{1}{7}$th of my money, I had 85*l*. left in my purse; What money had I at first?

Let x = money in my purse at first;

then $\frac{x}{4}+\frac{x}{7}$ = money paid away.

But money at first − money paid away = money remaining.

Hence $x-\left(\frac{x}{4}+\frac{x}{7}\right)= \quad 85$,

i. e. $x- \frac{x}{4}-\frac{x}{7} = \quad 85$.

Multiply by 4, then $4x - x - \frac{4x}{7} = 340$;

: . . . . . by 7 . . . $28x - 7x - 4x = 2380$,

$$\therefore\ 17x = 2380;\ \text{or}\ x = \frac{2380}{17} = 140l.$$

Question 7.

√ Of a battalion of soldiers (the officers being included), $\frac{3}{4}$ths are on duty, $\frac{1}{10}$th are sick, $\frac{3}{5}$ths of the remainder are absent, and there are 48 officers. What is the number of persons in the battalion?

Let x = the number of persons in the battalion.

Then $\frac{3}{4}$ths of x, or $\frac{3x}{4}$ = men on duty,

$\frac{1}{10}$th of x, or $\frac{x}{10}$ = the sick;

And $\frac{3x}{4} + \frac{x}{10}$, or $\frac{34x}{40} = \frac{17x}{20}$ = men on duty and sick.

Hence $x - \frac{17x}{20} = \frac{3x}{20}$ = remainder,

And $\frac{3}{5}$ths of $\frac{3x}{20}$, or $\frac{9x}{100}$, = $\frac{3}{5}$ths of remainder = the absent.

But the men on *duty*, the *sick*, the *absent*, and the officers, together make up the *whole battalion*;

i. e. $\frac{17x}{20} + \frac{9x}{100} + 48 = x$,

or $17x + \frac{9x}{5} + 960 = 20x$;

$\therefore\ 85x + 9x + 4800 = 100x$.

Hence $100x - 85x - 9x = 4800$,

or $6x = 4800$; or $x = \frac{4800}{6} = 800$.

Question 8.

There are two numbers, such, that 3 times the greater added to $\frac{1}{3}$d the lesser is equal to 36; and if twice the greater be subtracted from 6 times the lesser, and the remainder divided by 8, the quotient will be 4. What are the numbers?

Let x=the *greater* number,
y=the *lesser* number;

$$\text{Then }\left.\begin{aligned} 3x+\frac{y}{3}&=36 \\ \frac{6y-2x}{8}&=4 \end{aligned}\right\}\text{ or }\begin{aligned} 9x+y&=108 \\ 6y-2x&=32; \end{aligned}$$

$$\text{Or } y+9x=108\ (A).$$
$$6y-2x=32\ (B)$$

Multiply equation (A) by 6, then $6y+54x=648$
Subtract equation (B) . . . $6y-2x=32$;

then $56x=616$;

$$\therefore x=\frac{616}{56}=11.$$

From equation A . . . . $y=108-9x=108-99=9$.

Question 9.

There is a certain fraction, such, that if I add 3 to the numerator, its value will be $\frac{1}{3}$; and if I subtract *one* from the denominator, its [illegible]ue will be $\frac{1}{5}$. What is the fraction?

$$\left.\begin{aligned}\text{Let } x&=\text{its numerator}\\ y&=\text{denominator}\end{aligned}\right\}\text{ then the fraction is }\frac{x}{y}.$$

$$\left.\begin{aligned}&\text{Add 3 to the numerator, then } \frac{x+3}{y}=\frac{1}{3} \\ &\text{Subtract one from denom}^{r}\text{., and } \frac{x}{y-1}=\frac{1}{5}\end{aligned}\right\}\text{ or }\begin{aligned}3x+9&=y\\ 5x&=y-1.\end{aligned}$$

By *transposition*, $y-3x=9\ (A)$,
$y-5x=1\ (B)$.

Subtract equation (B) from (A), and we have

$$2x=8;$$

$$\therefore x=\frac{8}{2}=4 \text{ the numerator.}$$

From equation (A) $y=9+3x=9+12=21$ the denominator.

Hence the fraction required is $\frac{4}{21}$.

QUESTION 10.

A and *B* have certain sums of money; says *A* to *B*, give me 15*l.* of your money, and I shall have 5 times as much as you will have left; says *B* to *A*, give me 5*l.* of your money, and I shall have exactly as much as you will have left. What sum of money had each?

$$\left.\begin{array}{l}\text{Let } x=A\text{'s money} \\ \quad\; y=B\text{'s} \;\ldots\end{array}\right\}\text{then } x+15=\text{what } A \text{ would have, after receiving } 15l. \text{ from } B.$$

$$y-15=\text{what } B \text{ would have left.}$$

$$\text{Again } y+5=\text{what } B \text{ would have, after receiving } 5l. \text{ from } A.$$

$$x-5=\text{what } A \text{ would have left.}$$

$$\text{Hence, by the question, } \left.\begin{array}{l} x+15=5\times(y-15)=5y-75, \\ \text{and } y+5=x-5.\end{array}\right\}$$

$$\text{By transposition, } \left.\begin{array}{l} 5y-x=90 \ (A), \\ \text{and } y-x=-10 \ (B)\end{array}\right\}$$

$$\text{Subtract } (B) \text{ from } (A),\ 4y=100;$$

$$\therefore\ y=25=B\text{'s money.}$$

From equation (B), $x=y+10=25+10=35=A$'s money.

QUESTION 1[illegible]

A person bought a certain number of sheep for 94*l.*; having lost 7 of them, he sold $\frac{1}{4}$th of the remainder of them at *prime cost* for 20*l.* How many sheep had he at first?

Let x=number of sheep he had at first.

$$\text{Then } \frac{94}{x}=\frac{\text{whole sum}}{\text{number of sheep}}=\text{what } each \text{ sheep cost.}$$

Now $x-7$=number remaining, when 7 were lost;

$$\therefore\ \frac{x-7}{4}=\text{the number sold for } 20l.$$

But the *number sold* × *price of each*=whole price of sheep sold.

$$\text{Hence, by } substitution,\ \frac{x-7}{4}\times\frac{94}{x}=20,$$

$$\text{or } (x-7)\times 94=80x,$$

$$\text{i. e. } 94x-658=80x,$$

$$\text{or } 94x-80x=658,$$

$$\therefore\ 14x=658; \text{ or } x=\frac{658}{14}=47.$$

Question 12.

A and *B* have the *same* income; *A* is extravagant, and contracts an annual debt amounting to $\frac{1}{7}$th of it; *B* lives upon $\frac{4}{5}$ths of it; at the end of 10 years, *B* lends *A* money enough to pay off his debts, and has then 160*l.* to spare. What is their income?

Let x=their income.

Then $\frac{1}{7}$th of x, or $\frac{x}{7}$=*A*'s *annual* debt,

and $10\times\frac{x}{7}$ or $\frac{10x}{7}$=*A*'s debt contracted in 10 *years.*

As *B* lives upon $\frac{4}{5}$ths of his income, he saves annually $\frac{1}{5}$th of it;

hence, $\frac{x}{5}$=*B*'s *annual* saving,

and $10\times\frac{x}{5}$, or $\frac{10x}{5}$, or $2x$=*B*'s savings in 10 *years.*

But, by the question, *B*'s savings=*A*'s debt+160;

$$\therefore \text{ by substitution, } 2x=\frac{10x}{7}+160,$$

$$\text{or } 14x=10x+1120,$$

$$\text{and } 4x=1120\text{; or } x=\frac{1120}{}=280l.$$

Question 13.

A person was desirous of relieving a certain number of beggars by giving them 2*s.* 6*d.* each, but found that he had not money enough in his pocket by 3 shillings; he then gave them 2 shillings each, and had four shillings to spare. What money had he in his pocket; and how many beggars did he relieve?

Let x=money in his pocket (in *shillings*).
y=number of beggars.

Then $2\frac{1}{2}\times y$, or $\frac{5y}{2}$=N°. of *shill*^s^. which would have [been given at 2*s.* 6*d.* each.

and $2\times y$, or $2y$= . . . . at 2*s.* each.

Hence, by the question, $\frac{5y}{2}=x+3$ (A),

and $2y=x-4$ (B).

Subtract (B) from (A), then $\frac{y}{2}=7$, or $y=14$, the number of [beggars,

From eq^n. (B), $x=2y+4=28+4=32$ shillings in pocket.

QUESTION 14.

A person passed $\frac{1}{6}$th of his age in childhood, $\frac{1}{12}$th in youth, $\frac{1}{7}$th + 5 years in matrimony; he had *then* a son whom he survived 4 years, and who reached only $\frac{1}{2}$ the age of his father. At what age did this person die?

Let $x=$ age of the person at the time of his death.

Then $\frac{x}{6}=$ time spent in *childhood*.

$\frac{x}{12}= \ldots\ldots$ in *youth*.

$\frac{x}{7}+5= \ldots\ldots$ in *matrimony*.

$\therefore \frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5=$ age of the person when the son [was born,

and $x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-5=$ interval between the birth of the [son and the old man's death;

$\therefore x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-5-4=$ age of the son when he died.

But, by the question, the son died at $\frac{1}{2}$ the age of his father,

$$\text{Hence, } x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-9=\frac{x}{2}.$$

Multiply by 12, then $12x-2x-x-\frac{12x}{7}-108=6x$,

$$\text{or } 3x-\frac{12x}{7}=108,$$

$$\text{and } 21x-12x=756;$$

$$\therefore 9x=756; \text{ or } x=\frac{756}{9}=84.$$

QUESTION 15.

To find a number, such, that whether it is divided into *two* or *three* equal parts, the continued product of the parts shall be equal to the same quantity.

Let x=the number required.

Then $\frac{x}{2}\times\frac{x}{2}$=continued product, when the number [is divided into *two* parts.

and $\frac{x}{3}\times\frac{x}{3}\times\frac{x}{3}$=continued product, when the number [is divided into *three* parts.

Hence, by the question, $\frac{x}{2}\times\frac{x}{2}=\frac{x}{3}\times\frac{x}{3}\times\frac{x}{3}$, or $\frac{x^2}{4}=\frac{x^3}{27}$;

$\therefore 27x^2=4x^3$.

Divide by x^2, then $27=4x$,

and $x=\frac{27}{4}=6\frac{3}{4}$, the number required.

Qu. 16. There is a certain number, consisting of two digits. The *sum* of those digits is 5; and if 9 be added to the number itself, the digits will be inverted. What is the number?

Let x=*left-hand* digit.
y=*right-hand* digit.

Then by Art. 61. $10x+y$=the number itself,
and $10y+x$=the number with its digits *inverted.*

Hence, by the question, $x+y=5\,(A)$,
and $10x+y+9=10y+x$, or $9x-9y=-9$, or $x-y=-1(B)$.

Subtract (B) from (A), then $2y=6$, and $y=3$,

$x=5-y=5-3=2$;

$\therefore$ the number$=(10x+y)=23$.

Add 9 to this number, and it becomes 32, which is the number with the *digits inverted.*

Qu. 17. What two numbers are those whose difference is 10; and if 15 be added to their sum, the whole will be 43?

ANSWER, 9, and 19.

Qu. 18. There are two numbers whose difference is 14; and if 9 times the lesser be subtracted from 6 times the greater, the remainder will be 33. What are the numbers?

Answer. 17, and 31.

Qu. 19. What number is that, to which if I add 20, and from $\frac{2}{3}$ds of this sum I subtract 12, the remainder shall be 10? **Answ.** 13.

Qu. 20. What number is that, of which if I add $\frac{1}{3}$d, $\frac{1}{4}$th, and $\frac{2}{7}$ths together, the sum shall be 73?

Answ. 84.

Qu. 21. Two persons, *A* and *B*, lay out equal sums of money in trade; *A gains* 120*l.*, and *B loses* 80*l.*; and now *A*'s money is *treble* of *B*'s. What sum had each at first?

Answ. 180*l.*

Qu. 22. What number is that whose $\frac{1}{3}$d part exceeds its $\frac{1}{5}$th by 72? **Answ.** 540.

Qu. 23. There are two numbers whose sum is 37; and if 3 times the lesser be subtracted from 4 times the greater, and this difference divided by 6, the quotient will be 6. What are the numbers? **Answ.** 21, and 16.

Qu. 24. There are two numbers whose sum is 49; and if $\frac{1}{7}$th of the lesser be subtracted from $\frac{1}{5}$th of the greater, the remainder will be 5. What are the numbers?

Answ. 35, and 14.

Qu. 25. What two numbers are those, to one-third of the sum of which if I add 13, the result shall be 17; and if from half their difference I subtract *one*, the remainder shall be two? **Answ.** 9, and 3.

Qu. 26. There is a certain fraction, such, that if I add *one* to its numerator, it becomes $\frac{1}{2}$; if 3 be added to the denominator, it becomes $\frac{1}{3}$. What is the fraction?

Answ. $\frac{5}{12}$.

Qu. 27. A person has two horses, and a saddle worth 10*l.*; if the saddle be put on the *first* horse, his value becomes *double* that of the *second;* but if the saddle be put on the *second* horse, *his* value will not amount to that of the *first* horse by 13*l.* What is the value of each horse?

Answer, 56*l.*, and 33*l.*

Qu. 28. To divide the number 72 into three parts, so that $\frac{1}{2}$ the *first* part shall be equal to the *second*, and $\frac{3}{5}$ths of the *second* part equal to the third. Answ. 40, 20, and 12.

Qu. 29. A person after spending $\frac{1}{5}$th of his income *plus* 10*l.* had then remaining $\frac{1}{2}$ of it *plus* 35*l.* Required his income. Answ. 150*l.*

Qu. 30. A gamester at *one sitting* lost $\frac{1}{4}$th of his money, and then won 10 shillings; at a *second* he lost $\frac{1}{3}$d of the remainder, and then won 3 shillings; after which he had 3 guineas left. What money had he at first? Answ. 5*l.*

Qu. 31. There are two numbers, such, that $\frac{1}{2}$ the greater added to $\frac{1}{3}$d the lesser is 13; and if $\frac{1}{2}$ the lesser be taken from $\frac{1}{3}$d the greater, the remainder is nothing. What are the numbers? Answ. 18 and 12.

Qu. 32. There is a certain number, to the sum of whose digits if you add 7, the result will be three times the left-hand digit; and if from the number itself you subtract 18, the digits will be *inverted.* What is the number? Answ. 53.

Qu. 33. Divide the number 90 into four such parts, that the first *increased* by 2, the second *diminished* by 2, the third *multiplied* by 2, and the fourth *divided* by 2, may all be equal to the same quantity. Answ. 18, 22, 10, 40.

Qu. 34. A merchant has two kinds of tea, one worth 9*s.* 6*d.* per pound, the other 13*s.* 6*d.* How many pounds of each must he take to form a chest of 104lbs. which shall be worth 56*l.*? Answ. 33 at 13*s.* 6*d.*

71 at 9*s.* 6*d.*

Qu. 35. A vessel containing 120 gallons is filled in 10 minutes by two spouts running *successively*; the one runs 14 gallons in a minute, the other 9 gallons in a minute. For what time has each spout run?

Answer, 14-gallon spout runs 6 minutes.
9-gallon spout . . . 4 minutes.

Qu. 36. In the composition of a certain number of pounds of gunpowder, $\frac{2}{3}$ds the whole + 10 was *nitre*; $\frac{1}{6}$th the whole − $4\frac{1}{2}$ was *sulphur*; and the charcoal was $\frac{1}{7}$th of the *nitre* − 2. How many pounds of gunpowder were there?

Answ. 69 pounds.

Qu. 37. To find three numbers, such, that the *first* with $\frac{1}{2}$ the sum of the *second* and *third* shall be 120; the *second* with $\frac{1}{5}$th the difference of the *third* and *first* shall be 70; and $\frac{1}{2}$ the sum of the three numbers shall be 95.

Answ. 50, 65, 75.

CHAP. V.

ON QUADRATIC EQUATIONS.

Quadratic Equations are divided into *pure* and *adfected*. *Pure* quadratic equations are those which contain only the *square* of the unknown quantity; such as $x^2=36$; $x^2+5=54$; $ax^2-b=c$; &c. *Adfected* quadratic equations are those which involve both the *square* and *simple power* of the unknown quantity, such as $x^2+4x=45$; $3x^2-2x=21$; $ax^2+2bx=c+d$; &c. &c.

XX.

On the Solution of Pure Quadratic Equations.

77. The Rule for the solution of pure quadratic equations is this; "Transpose the terms of the equation in such a "manner, that those which contain x^2 may be on one side "of the equation, and the *known quantities* on the other; "divide (if necessary) by the coefficient of x^2; then extract "the square root of each side of the equation, and it will "give the value of x."

EXAMPLE 1.

Let $x^2 + 5 = 54$.

By transposition, $x^2 = 54 - 5 = 49$.

Extract the square root of both sides of the equation . . . . . } then $x = \sqrt{49} = 7$.

Ex. 2.

Let $3x^2 - 4 = 71$.

By transposition, $3x^2 = 71 + 4 = 75$.

Divide by 3, $x^2 = \frac{75}{3} = 25$.

Extract the square root, $x = \sqrt{25} = 5$

Ex. 3.

Let $5x - 27 = 3x + 215$.

By transposition, $5x - 3x^2 = 215 + 27$,

or $2x^2 = 242$;

$\therefore x^2 = \frac{242}{2} = 121$, and $x = 11$.

Ex. 4.

Let $ax^2 - b = c$;

then $ax^2 = c + b$,

and $x = \frac{c+b}{a}$, or $x = \sqrt{\frac{c+b}{a}}$.

Ex. 5.

Let $ax^2 - 5c = bx - 3c + d$.

Then $ax^2 - bx = 5c - 3c + d$,

or $(a - b)\, x^2 = 2c + d$;

$\therefore x^2 = \frac{2c+d}{a-b}$, and $x = \sqrt{\frac{2c+d}{a-b}}$.

Ex. 6. $5x - 1 = 244$ . . ANSWER, $x = 7$

Ex. 7. $9x^2 + 9 = 3x^2 + 63$ . . . . . . $x = 3$.

Ex. 8. $\frac{4x + 5}{9} = 45$ . . . . $x = 10$.

Ex. 9. $bx^2+c+3=2bx^2+1$ . . . . $x=\sqrt{\frac{c+2}{b}}$

Ex. 10. $2ax^2+b-4=cx^2-5+d-ax^2$ . . $x=\sqrt{\frac{d-b-1}{3a-c}}$.

XXI.

On the Solution of Adfected Quadratic Equations.

78. The most general form under which an adfected quadratic equation can be exhibited is $ax^2+bx=c$; where a, b, c may be any quantities whatever, *positive* or *negative*, *integral* or *fractional*. Divide each side of this equation by a, then $x^2+\frac{b}{a}x=\frac{c}{a}$. Let $\frac{b}{a}=p$, $\frac{c}{a}=q$; then this equation is reduced to the form $x^2+px=q$, where p and q may be any quantities whatever, positive or negative, integral or fractional.

79. From the twofold form under which adfected quadratic equations may be expressed, there arise two Rules for their solution.

RULE I.

Let $x^2+px=q$.

Add $\frac{p^2}{4}$ to each side of the equation, then $x^2+px+\frac{p^2}{4}=\frac{p^2}{4}+q=\frac{p^2+4q}{4}$.

Extract the square root of each side of the equation, then $x+\frac{p}{2}=\frac{\pm\sqrt{p^2+4q}}{2}$ (a)

and $x=\frac{\pm\sqrt{p^2+4q}-p}{2}$.

Hence it appears, that "if to each side of the equation "there be added the *square of half the coefficient of the* "*second term*, there will arise, on the left-hand side of the "equation, a quantity which is the square of $x+\frac{p}{2}$; and by

---

(a) Since the square of $+a$ is $+a^2$, and of $-a$ is *also* $+a^2$, the *square root* of $+a^2$ may be either $+a$ or $-a$; hence the square root of p^2+4q may be expressed by $\pm\sqrt{p^2+4q}$.

" extracting the square root of each side of the resulting " equation, we obtain a *simple* equation; from which the " value of x may be determined."

RULE II.

Let $ax^2 + bx = c$.

Multiply each side of the equation by $4a$, then $4a^2x^2 + 4abx = 4ac$.

Add b^2 to each side, we have $4a^2x^2 + 4abx + b^2 = 4ac + b^2$.

Extract the square root as before, $2ax + b = \pm\sqrt{4ac + b^2}$

$$\therefore\ 2ax = \pm\sqrt{4ac + b^2} - b$$

$$\text{and } x = \frac{\pm\sqrt{4ac + b^2} - b}{2a}$$

From which we infer, that " if each side of the equation " be multiplied by *four times the coefficient of* x^2, and to each " side there be added *the square of the coefficient of* x, the " quantity on the left-hand side of the equation will be the " square of $2ax + b$. Extract the square root of each side " of the equation, and there arises a *simple equation*, from " which the value of x may be determined(b)."

If $a = 1$, the equation is reduced to the form $x^2 + px = q$; in this case, therefore, the Rule may be applied, by " multiplying each side of the equation by 4, and adding the " square of the coefficient of x."

80. Either of these Rules may of course be applied to the solution of adfected quadratic equations; but it may be proper to observe, that in equations of the form $ax^2 + bx = c$ where a is a *small* number, and in those of the form $x^2 + px = q$ where p is an *odd* number, RULE II. will be found by far the most convenient.

81. From the form in which the value of x is exhibited in each of these Rules, it is evident that it will have *two* values;

(b) The principle of this Rule will be found in the *Bija Ganita*, a *Hindoo* Treatise on the Elements of Algebra. See Mr. COLEBROOKE's Translation of this very curious work.

one corresponding to the sign +, and the other to the sign −, of the radical quantity. In the following examples, the *positive* values only of x are inserted at the end of the solution.

EXAMPLE 1.

Let $x^2+8x=65$.

By RULE I. add the square of 4 $\left(i.e.\ \frac{p^2}{4}\right)$ to each side of the equation, then . . ∴ $x^2+8x+16=65+16=81$.

Extract the square root of each side of the equation, then $\left(x+\frac{p}{2}, \text{ or}\right) x+4=\sqrt{81}=9$, and $x=5$.

Ex. 2.

Let $x^2-4x=45$

By RULE I. add the square of 2, *i.e.* 4; then . . . $x^2-4x+4=45+4=49$.

Extract the square root, and $\left(x-\frac{p}{2}, \text{ or}\right) x-2=\sqrt{49}=7$, and $x=9$.

Ex. 3.

Let $3x^2+5x=42$.

By Rule II. multiply each side of the equation by ($4a$) 12; then . . . . . $36x^2+60x=504$.

Add (b^2) 25 to each side of the equation, we have $36x^2+60x+25=504+25=529$.

Extract the square root of each side of the equation, which gives $(2ax+b, \text{ or})\ 6x+5=\sqrt{529}=23$,

$\therefore 6x=18$, and $x=3$.

Ex. 4.

Let $7x^2-20x=32$,

$$\therefore x^2-\frac{20x}{7}=\frac{32}{7}.$$

Complete the Square by RULE I.

$$\text{then } x^2-\frac{20x}{7}+\frac{100}{49}=\frac{32}{7}+\frac{100}{49}=\frac{224}{49}+\frac{100}{49}=\frac{324}{49}.$$

$$\text{Hence } x-\frac{10}{7}=\sqrt{\frac{324}{49}}=\frac{18}{7}, \text{ and } x=\frac{28}{7}=4.$$

Ex. 5.

Let $x^2-15x=-54$.

By Rule II. multiply by 4 . . . then $4x^2-60x=-216$.

Add (b^2) 225 to each side . . and $4x^2-60x+225=225-216=9$.

Extract the square root, $2x-15=\pm\sqrt{9}=\pm 3$

$\therefore\ 2x=15\pm 3=18$ or 12, and $x=9$ or 6.

Ex. 6.

Let $4x^2-3x=85$.

By Rule II. multiply by 16, and add the square of 3 to each side of the equation $64x^2-48x+9=1360+9=1369$.

Extract the square root $8x-3=\sqrt{1369}=37$, or $x=\frac{40}{8}=5$.

Ex. 7.

Let $\frac{4x^2}{3}-11=\frac{x}{3}$.

Multiply by 3, then $4x^2-33=x$.

By transposition . . $4x^2-x=33$.

Multiply by 16, and add 1 to each side of the equation (Rule II.) $64x^2-16x+1=528+1=529$.

Extract the square root, $8x-1=\sqrt{529}=23$, or $x=\frac{24}{8}=3$.

Ex. 8.

Let $5x^2+4x=273$;

then $x^2+\frac{4x}{5}=\frac{273}{5}$

and by Rule I. $x^2+\frac{4x}{5}+\frac{4}{25}=\frac{273}{5}+\frac{4}{25}=\frac{1369}{25}$

$\therefore\ x+\frac{2}{5}=\sqrt{\frac{1369}{25}}=\frac{37}{5}$

and $x=\frac{37}{5}-\frac{2}{5}=\frac{35}{5}=7$.

Ex. 9.

Let $\frac{7}{x+1}+\frac{2}{x}=5.$

Mult^y. by $x+1$, then $7+\frac{2x+2}{x}=5x+5.$

. . . . by x . . . . $7x+2x+2=5x^2+5x.$

By transposition, $5x^2-4x=2.$

By Rule II. $100x^2-80x+16=40+16=56.$

Extract the square root, $10x-4=\sqrt{56}$

and $10x=\sqrt{56}+4=7.48+4=11.48$

$\therefore x=\frac{11.48}{10}=1.148.$

Ex. 10.

Let $13x^2+2x=60.$

Divide by 13, $x+\frac{2x}{13}=\frac{60}{13}.$

By Rule I. add the square of $\frac{1}{13}$, $x^2+\frac{2x}{13}+\frac{1}{169}=\frac{60}{13}+\frac{1}{169}=\frac{780}{169}+\frac{1}{169}=\frac{781}{169}.$

Extract the square root $x+\frac{1}{13}=\frac{\sqrt{781}}{13}=\frac{27.94}{13}$

$\therefore x=\frac{27.94-1}{13}=\frac{26.94}{13}=2.07.$

Ex. 11.

Let $2bx^2-cx=d.$

By Rule II. multiply by $8b$, and add c^2 $16b^2x^2-8bcx+c^2=8bd+c^2.$

Extract the square root $4bx-c=\sqrt{8bd+c^2}$ or $x=\frac{\sqrt{8bd+c^2}+c}{4b}.$

Ex. 12. $x^2+12x=108$ . . . . $x=6.$

Ex. 13. $x^2-14x=51$ . . . . $x=17.$

Ex. 14. $x^2+6bx=c$ . . . . $x=\sqrt{c+9b^2}-3b.$

Ex. 15. $3x^2+2x=161$ . . . . $x=7.$

Ex. 16. $2x^2-5x=117$ . . . . $x=9.$

Ex. 17. $3x^2-2x=280$ . . . . $x=10$.

Ex. 18. $5x^2+4x=273$ . . . . $x=7$.

Ex. 19. $4x^2-7x=492$ . . . . $x=12$.

Ex. 20. $\frac{x^2}{6}-1=x+11$ . . . . $x=12$.

Ex. 21. $\frac{2x}{3}+\frac{1}{x}=\frac{7}{3}$ . . . . $x=3$ or $\frac{1}{2}$.

Ex. 22. $\frac{x^2}{3}-\frac{x}{2}=9$ . . . . $x=6$.

Ex. 23. $\frac{6}{x+1}+\frac{2}{x}=3$ . . . . $x=2$.

Ex. 24. $x^2-34=\frac{1}{3}x$ . . . . $x=6$.

Ex. 25. $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}$ . . . . $x=25$ or 1.

Ex. 26. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$ . . . . $x=2$.

Ex. 27. $\frac{3x}{x+2}-\frac{x-1}{6}=x-9$ . . . . . . $x=10$.

Ex. 28. $x^2-6x+19=13$ . . . . $x=4.732$ or 1.268.

Ex. 29. $5x^2+4x=25$ . . . . $x=1.871$.

Ex. 30. $4ax^2-bx=c$ . . . . $x=\frac{b+\sqrt{b^2+16ac}}{8a}$.

Ex. 31. $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$ . . . . $x=1\pm\sqrt{1-a^2}$.

XXII.

On the Solution of Questions producing Quadratic Equations.

82. In the solution of Questions which involve Quadratic Equations, sometimes *both*, and sometimes only one of the values of the unknown quantity will answer the conditions required. This is a circumstance which may always be very readily determined by the nature of the question itself.

QUESTION 1.

To divide the number 56 into two such parts, that their product shall be 640.

Let x = *one* part,

then $56-x$ = the *other* part,

and $x(56-x)$ = *product* of the two parts.

Hence, by the question, $x(56-x)=640$,

or $56x-x^2=640$.

By transposition, $x^2-56x=-640$.

By completing the square, RULE I. $x^2-56x+784=784-640=144$;

$\therefore x-28=\pm 12$, and $x=40$ or 16.

In this case, it appears that the *two* values of the unknown quantity are the *two* parts into which the given number was required to be divided.

QUESTION 2.

There are two numbers whose difference is 7, and half their product *plus* 30 is equal to the square of the *lesser* number. What are the numbers?

Let x = the *lesser* number,

then $x+7$ = the *greater* number,

and $\frac{x\times(x+7)}{2}+30$ = half their product *plus* 30.

Hence, by the question, $\frac{x(x+7)}{2}+30=x^2$ (square of *lesser*),

or $\frac{x^2+7x}{2}+30=x^2$.

Multiply by 2 . . . $x^2+7x+60=2x^2$.

By transposition . . . . . $x^2-7x=60$.

Multiply by 4, and add 49 (Rule II.) $4x^2-28x+49=240+49=289$,

$\therefore 2x-7=17$

$2x=24$, or $x=12$ = *lesser* number;

hence $x+7=19$ = *greater* number.

QUESTION 3.

To divide the number 30 into two such parts, that their product may be equal to *eight* times their difference.

Let x = the *lesser* part,
then $30-x$ = the *greater* part,
and $30-x-x$ or $30-2x$ = their *difference*.

Hence, by the question, $x(30-x)=8(30-2x)$,
or $30x-x^2=240-16x$.

By transposition, $x^2-46x=-240$.

Complete the square, (RULE I.) $x^2-46x+529=529-240=289$;

$\therefore x-23=\pm 17$,
and $x=23\pm 17=40$ or 6 = *lesser* part;
$30-x=30-6=24$ = *greater* part.

In this case, the solution of the equation gives 40 and 6 for the *lesser* part. Now, as 40 cannot possibly be a *part* of 30, we take 6 for the *lesser* part, which gives 24 for the *greater* part; and the two numbers, 24 and 6, answer the conditions required.

QUESTION 4.

A person bought cloth for 33*l.* 15*s.* which he sold again at 2*l.* 8*s.* per piece, and gained by the bargain as much as one piece cost him; Required the number of pieces.

Let x = the number of pieces.

Then $\frac{675}{x}$ = number of *shillings* each piece *cost*,

and $48x$ = number of *shillings* he *sold* the whole for;

$\therefore 48x-675$ = what he gained by the bargain.

Hence, by the question, $48x-675=\frac{675}{x}$.

By transposition and division, $x^2-\frac{225}{16}x=\frac{225}{16}$.

Complete the square (RULE I.) $\left.\right\}$ $x^2-\frac{225}{16}x+\frac{50625}{1024}=\frac{225}{16}+\frac{50625}{1024}=\frac{65025}{1024}$;

$$\therefore x-\frac{225}{32}=\frac{225}{32}, \text{ and } x=\frac{480}{32}=15.$$

QUESTION 5.

A and *B* set off at the *same time* to a place at the distance of 150 miles. *A* travels 3 miles an hour *faster* than *B*, and arrives at his journey's end 8 hours and 20 minutes *before* him. At what rate did each person travel per hour?

Let x=rate per hour at which *B* travels.

Then $x+3$= . . . . *A* . . . .

And $\frac{150}{x}$=number of hours for which *B* travels.

$\frac{150}{x+3}$= . . . . *A* . . . .

But *A* is 8 hours 20 minutes ($8\frac{1}{3}$ hours) *sooner* at his journey's end than *B*;

$$\text{Hence } \frac{150}{x+3}+8\tfrac{1}{3}=\frac{150}{x},$$

$$\text{or } \frac{150}{x+3}+\frac{25}{3}=\frac{150}{x}$$

By reduction, $x^2+3x=54.$

Complete the square, $x^2+3x+\frac{9}{4}=54+\frac{9}{4}=\frac{225}{4}$ (Rule I.);

$$\therefore x+\frac{3}{2}=\frac{15}{2};$$

and $x=\frac{15-3}{2}=6$ miles an hour for *B*

$x+3=9$ . . . . for *A*.

QUESTION 6.

Some bees had alighted upon a tree; at one flight the square root of half of them went away; at another $\frac{8}{9}$ths of them; two bees then remained. How many then alighted on the tree? (a)

(a) This question, and the mode of solution, is taken from the *Bija Ganita*.

Let $2x^2$=the N° of bees;

$$\text{then } x+\frac{16x^2}{9}+2=2x^2,$$

$$\text{or } 9x+16x^2+18=18x^2;$$

$$\therefore 18x^2-16x^2-9x=18,$$

$$\text{or } 2x^2-9x=18.$$

(Rule II.) Multiply by 8,

$$16x^2-72x=144.$$

Add 81; then $16x^2-72x+81=225$,

$$\text{or } 4x-9=15;$$

$\therefore 4x=15+9=24$, and $x=6$, $\therefore 2x^2=72$=N° of bees.

Qu. 7. To divide the number 33 into two such parts, that their product shall be 162. Answer, 27 and 6.

Qu. 8. What two numbers are those whose sum is 29, and product 100? Answ. 25 and 4.

Qu. 9. The difference of two numbers is 5, and $\frac{1}{4}$th part of their product is 26. What are the numbers? Answ. 13 and 8.

Qu. 10. The difference of two numbers is 6; and if 17 be added to *twice the square of the lesser*, it will be equal to the *square of the greater*. What are the numbers? Answ. 17 and 11, or 7 and 1.

Qu. 11. There are two numbers whose sum is 30; and $\frac{1}{3}$d of their product *plus* 18 is equal to the square of the *lesser* number. What are the numbers? Answ. 21 and 9.

Qu. 12. There are two numbers whose product is 120. If 2 be added to the lesser, and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers? Answ. 15 and 8.

Qu. 13. *A* and *B* distribute 1200*l.* each among a certain number of persons. *A* relieves 40 persons more than *B*, and *B* gives 5*l.* apiece to each person *more* than *A*. How many persons were relieved by *A* and *B* respectively? Answ. 120 by *A*, 80 by *B*.

Qu. 14. A person bought a certain number of sheep for 120*l.* If there had been 8 more, each sheep would have cost him 10 shillings less. How many sheep were there? Answ. 40.

Qu. 15. A person bought a certain number of sheep for 57*l.* Having lost 8 of them, and sold the remainder at 8 shillings a-head profit, he is no loser by the bargain. How many sheep did he buy? Answ. 38.

Qu. 16. *A* and *B* set off at the same time to a place at the distance of 300 miles. *A* travels at the rate of one mile an hour faster than *B*, and arrives at his journey's end 10 hours before him. At what rate did each person travel per *hour*? Answ. *A* travelled 6 miles per hour.
B . . . . . 5 . . . .

XXIII.

On Quadratic Equations having Impossible Roots.

83. In the solution of the adfected Quadratic Equation $x^2+px=q$ (Art. 79.) the two values of x were shewn to be equal to $\frac{\pm\sqrt{p^2+4q}-p}{2}$. If q be a *negative* quantity, and p^2 less than $4q$, then the quantity p^2-4q is negative, and consequently the quantity $\pm\sqrt{p^2-4q}$ comes under the description of the radical quantities mentioned in Art 56. In this case, the two roots, or values of x, are said to be *impossible*.

Example 1.

Let $x^2+8x+31=0$, or $x^2+8x=-31$.

Complete the square, (Rule I.)

$$\text{then } x^2+8x+16=-31+16=-15,$$

$$\text{and } x+4=\pm\sqrt{-15}, \text{ or } x=-4\pm\sqrt{-15}.$$

Ex. 2.

Let $x^2-12x+50=0$, or $x^2-12x=-50$.

Complete the square, (Rule I.)

$$x^2-12x+36=-50+36=-14,$$

$$\text{and } x-6=\pm\sqrt{-14};\ \therefore\ x=6\pm\sqrt{-14}.$$

Ex. 3.

To divide the number 16 into two such parts, that their product shall be equal to 70.

Let x=one part,

then $16-x$=the other part.

Hence $x\,(16-x)$ or $16x-x^2=70$.

Transpose, and $x^2-16x=-70$.

Complete the square,

$$x^2-16x+64=-70+64=-6,$$

$$\therefore\ x-8=\pm\sqrt{-6},\text{ or } x=8\pm\sqrt{-6}.^{(a)}$$

Ex. 4. $2x^2+15=3x$ . . . . . $x=\dfrac{3\pm\sqrt{-111}}{4}$

Ex. 5. $3x-\frac{1}{4}x^2=10$ . . . . . $x=6\pm\sqrt{-4}$.

Ex. 6. To divide the number 20 into two such parts, that their product shall be 105. $x=10\pm\sqrt{-5}$.

XXIV.

On the Solution of Quadratic Equations of the form $x^{2n}+px^n=q$.

84. Let $y=x^n$, then (by Case III. Art. 65.) $y^2=x^{2n}$; and substituting these values for x^{2n} and x^n in the equation $x^{2n}+px^n=q$, it is transformed into $y^2+py=q$, where the value of y may be determined by the foregoing Rules. Having the value of y, the value of x may be found; for $x^n=y$, $\therefore x=\sqrt[n]{y}$. We are thus enabled to solve equations

---

(a) It is very well known that the *greatest* product which can arise from the multiplication of the two parts into which any given number may be divided, is when these parts are *equal:* the greatest product, therefore, which could arise from the division of the number 16 into two parts, is when each of them is 8; hence, in requiring "to divide the number 16 into two such parts that their product shall be 70," the solution of the question is *impossible.*

in which the unknown quantity is found only in *two* terms, and where the index of the highest power is *double* the index of the lowest, like common quadratics.

EXAMPLE 1.

Let $x^4-6x^2=27$.

$$\left.\begin{array}{r}\text{If } x^2=y,\\ \text{then } x^4=y^2,\end{array}\right\}\quad \therefore\ y^2-6y=27.$$

By RULE I. $y^2-6y+9=27+9=36$,

and $y-3=6$, or $y=9$.

But since $x^2=y$, $x=\sqrt{y}$, $\therefore\ x=\sqrt{9}=3$.

Ex. 2.

Let $x^6-2x^3=48$.

These equations are often solved by the common Rules. without the formality of substitution; thus,

Complete the square, (RULE I.) $x^6-2x^3+1=48+1=49$

Extract the root, $x^3-1=7$, $\therefore\ x^3=8$, and $x=\sqrt[3]{8}=2$.

Ex. 3.

Let $2x-7\sqrt{x}=99$.

$$\left.\begin{array}{r}\text{Put } y^2=x\\ \text{then } y=\sqrt{x}\end{array}\right\}\quad \therefore\ 2y^2-7y=99.$$

By Rule II. $16y^2-56y+49=792+49=841$,

and $4y-7=29$,

or $4y=36$, and $y=9$; $\therefore\ x=y^2=81$.

Ex. 4.

To resolve the number *a* into two such factors, that the sum of their *n*th powers shall be equal to *b*.

Let $x=$ one factor,

then $\dfrac{a}{x}=$ the other factor.

Hence $x^n+\dfrac{a^n}{x^n}=b$, or $x^{2n}+a^n=bx^n$, $\therefore\ x^{2n}-bx^n=-a^n$

By RULE II. $4x^{2n}-4bx^n+b^2=b^2-4a^n$,

and $2x^n-b=\pm\sqrt{b^2-4a^n}$, or $2x^n=b\pm\sqrt{b^2-4a^n}$,

and $x^n=\dfrac{b\pm\sqrt{b^2-4a^n}}{2}$. $\therefore\ x=\sqrt[n]{\dfrac{b\pm\sqrt{b^2-4a^n}}{2}}$.

The two values of *x* are the two factors required.

Ex. 5. $x^4+4x^2=12$ . . . . $x=\sqrt{2}$.

Ex. 6. $x^6-8x^3=513$ . . . . $x=3$.

Ex. 7. $2x^4-x^2=496$ . . . . $x=4$.

Ex. 8. To resolve the number 18 into two such factors, that the sum of their *cubes* shall be 243. (See Ex. 4.)

ANSWER, 6 and 3.

XXV.

On the Solution of Quadratic Equations containing Two unknown Quantities.

The solution of equations with *two* unknown quantities, in which one or both these quantities are found in a quadratic form can only, in *particular cases*(b), be effected by means of the preceding Rules. Of these cases, the two following are very well known.

CASE I.

85. "When one of the equations by which the values "of the unknown quantities are to be determined, is a "*simple* equation;" in which case, the Rule is, "to find a "value of one of the unknown quantities from that simple "equation, and then **substitute** for it the value so found, "in the other equation. The resulting equation will be "a quadratic. which may be solved by the ordinary Rules." Thus,

Let $\left.\begin{array}{l} ax+by=c \\ a'x^2+b'xy+c'y^2=d \end{array}\right\}$ be the two equations, in which the values of x and y are to be determined.

From the 1st equation $x=\dfrac{c-by}{a}$.

Substitute

(b) The most complete form under which quadratic equations containing two unknown quantities could be expressed, is this,

$$ax^2+by^2+cxy+dx+ey=m$$
$$a'x^2+b'y^2+c'xy+d'x+e'y=m';$$ but the general solution of these equations can only be effected by means of equations of higher dimensions than quadratics.

Substitute this for x in the 2d equation, $\Big\}$ then $a'\left(\frac{c-by}{a}\right)^2+b'\left(\frac{cy-by^2}{a}\right)+c'y^2=d,$

$$\text{or } \frac{a'c^2-2a'bcy+a'b^2y^2}{a^2}+\frac{b'cy-bb'y^2}{a}+c'y^2=d,$$

which reduced, is

$$(a'b^2-abb'+a^2c')y^2+(ab'c-2a'bc)y=a^2d-a'c^2,$$

a common quadratic equation, from which the value of y may be found.

EXAMPLE I.

$$\left.\begin{array}{r}\text{Let } x+2y=7,\\ \text{and } x^2+3xy-y^2=23\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

From 1st equation, $x=7-2y$, $\therefore x^2=49-28y+4y^2$;
Substitute these values for x and x^2 in the 2d equation,
then $49-28y+4y^2+21y-6y^2-y^2=23$,

$$\text{or } 3y^2+7y=49-23=26.$$

By RULE II. $36y^2+84y+49=312+49=361$,

$$\therefore 6y+7=19, \text{ or } 6y=12, \text{ and } y=2;$$

$$\therefore x=7-2y=7-4=3.$$

Ex. 2.

$$\left.\begin{array}{l}\text{Let } \frac{2x+y}{3}=9\\ \text{and } 3xy=210\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

From 1st equation, $2x+y=27$;

$$\therefore 2x=27-y, \text{ and } x=\frac{27-y}{2}.$$

$$\text{Hence, } 3xy=3\times\frac{27-y}{2}\times y=210,$$

$$\text{or } 3\times(27-y)\times y=420$$

$$81y-3y^2=420$$

$$27y-\ y^2=140;$$

$$\text{or } y^2-27y=-140.$$

By RULE II. $4y^2-108y+729=729-560=169$;

$$\therefore 2y-27=\pm 13, \text{ or } y=\frac{27\pm 13}{2}=20 \text{ or } 7,$$

$$\text{and } x=\frac{27-20 \text{ or } 7}{2}=\frac{7 \text{ or } 20}{2}=3\tfrac{1}{2} \text{ or } 10.$$

Ex. 3.

There is a certain number consisting of two digits. The left-hand digit is equal to 3 times the right-hand digit; and if 12 be subtracted from the number itself, the remainder will be equal to the square of the left-hand digit. What is the number?

Let x be the left-hand digit, and y the other; $\Big\}$ then, by Art. 61, $10x+y$ is the number.

$$\left.\begin{array}{r}\text{Hence, } x=3y\\ \text{and } 10x+y-12=x^2\end{array}\right\}\text{by the question;}$$

$$\left.\begin{array}{r}\therefore\text{ by sub-}\\ \text{stitution}\end{array}\right\}30y+y-12=9y^2,\ (\text{for } 10x=30y, \text{ and } x^2=9y^2);$$

$$9y^2-31y=-12;$$

$$\therefore\ y^2-\frac{31}{9}y=-\frac{12}{9}.$$

By Rule I. $y^2-\frac{31}{9}y+\frac{961}{324}=\frac{961}{324}-\frac{12}{9}=\frac{961-432}{324}=\frac{529}{324}.$

$$\text{Hence, } y-\frac{31}{18}=\frac{23}{18};\ \text{or } y=\frac{54}{18}=3,$$

$x=3y=9$; and consequently the number is 93.

Ex. 4. Let $\left.\begin{array}{r}2x-3y=1\\ 2x^2+xy-5y^2=20\end{array}\right\}$ to find the values of x and y.

Answer, $x=5$, $y=3$.

Ex. 5. There are two numbers, such, that if the lesser be taken from three times the greater, the remainder will be 35; and if four times the greater be divided by three times the lesser *plus* one, the quotient will be equal to the lesser number. What are the numbers? Answ. 13 and 4.

Ex. 6. What number is that, the *sum* of whose digits is 15; and if 31 be added to their *product*, the digits will be inverted? Answ. 78.

Case II.

86. When x^2, y^2, or xy, is found in every term of the two equations, they assume the form of

$$ax^2+bxy+cy^2=d,$$

$$a'x^2+b'xy+c'y^2=d';$$ and their solution may be effected in the following manner.

Assume $x=vy$, then $x^2=v^2y^2$; substitute these values for x^2 and x in both equations, then we have

$$av^2y^2+bvy^2+cy^2=d, \text{ or } y^2=\frac{d}{av^2+bv+c} \quad (A)$$

$$a'v^2y^2+b'vy^2+c'y^2=d', \text{ or } y^2=\frac{d'}{a'v^2+b'v+c'} \quad (B)$$

$$\text{Hence } \frac{d}{av^2+bv+c}=\frac{d'}{a'v^2+b'v+c'}$$

or $(a'd-ad')v^2+(b'd-bd')v=cd'-c'd$; which is a quadratic equation, from whence the value of v may be determined. Having the value of v, the value of y may be found from either of the equations (A) or (B); and then the value of x, from the equation $x=vy$.

Example 1.

$$\text{Let } 2x^2+3xy+y^2=20$$
$$5x^2+4y^2=41;$$

Assume $x=vy$, then $2v^2y^2+3vy^2+y^2=20$, or $y^2=\dfrac{20}{2v^2+3v+1}$,

and $5v^2y^2+4y^2=41$, or $y^2=\dfrac{41}{5v^2+4}$;

$$\text{Hence } \frac{20}{2v^2+3v+1}=\frac{41}{5v^2+4},$$

which reduced, is, $6v^2-41v=-13$;

$$\therefore v^2-\frac{41v}{6}=-\frac{13}{6}.$$

By Rule I. $v^2-\dfrac{41v}{6}+\dfrac{1681}{144}=\dfrac{1369}{144}$;

$$\therefore v-\frac{41}{12}=\frac{\pm 37}{12}; \text{ or } v=\frac{41\pm 37}{12}=\frac{13}{2} \text{ or } \frac{1}{3}.$$

Let $v=\dfrac{1}{3}$, then $y^2=\dfrac{41}{5v^2+4}=\dfrac{41}{\frac{5}{9}+4}=\dfrac{369}{41}=9$, or $y=3$,

$$x=vy=\tfrac{1}{3}\times 3=1.$$

Ex. 2.

What two numbers are those, whose sum multiplied by the greater is 77; and whose difference multiplied by the lesser is equal to 12?

Let x=greater number,

y=lesser.

Then $(x+y)\times x=x^2+xy=77$,

and $(x-y)\times y=xy-y^2=12$.

Assume $x=vy$;

Then $v^2y^2+vy^2=77$, or $y^2=\frac{77}{v^2+v}$;

and $vy^2-y^2=12$, or $y^2=\frac{12}{v-1}$.

Hence, $\frac{12}{v-1}=\frac{77}{v^2+v}$,

or $12v^2+12v=77v-77$;

which gives $v^2-\frac{65}{12}v=-\frac{77}{12}$,

and $v^2-\frac{65}{12}v+\frac{4225}{576}=\frac{529}{576}$;

$$\therefore v=\frac{65\pm 23}{24}=\frac{88 \text{ or } 42}{24}=\frac{11}{3} \text{ or } \frac{7}{4}.$$

Either value of v will answer the conditions of the question; but take $v=\frac{7}{4}$; then $y^2=\frac{12}{v-1}=\frac{12}{\frac{7}{4}-1}=\frac{48}{7-4}=\frac{48}{3}=16$.

and $y=4$, $\therefore x=vy=\frac{7}{4}\times 4=7$.

Hence, the numbers are 4 and 7.

Ex. 3. Find two numbers, such, that the square of the greater *minus* the square of the lesser may be 56; and the square of the lesser *plus* $\frac{1}{3}$d their product may be 40.

ANSWER, 9 and 5.

Ex. 4. There are two numbers, such, that 3 times the square of the greater *plus* twice the square of the lesser is 110; and half their product *plus* the square of the lesser is 1. What are the numbers? ANSW. 6 and 1.(a)

(a) For a great variety of questions relating to quadratic equations which contain two unknown quantities, see BLAND's *Algebraical Problems*, 1812.

XXVI.

On the Solution of certain Equations, in which the Two unknown Quantities (x and y) are similarly involved.

87. Let x and y be any two numbers, of which x is the greater, and y the lesser; let $x+y=2s$, $x-y=2z$; then, by Art. 28, $x=s+z$, and $y=s-z$. Now, let $x^2+y^2=a$, $x^3+y^3=b$, $x^4+y^4=c$, and $x^5+y^5=d$; then the values of x and y may be found in terms of the known quantities s, a, b, [illegible] d, in the following manner.

I. $x^2=(s+z)^2=s^2+2sz+z^2$

$y^2=(s-z)^2=s^2-2sz+z^2$;

$\therefore$ by addition,

$$x^2+y^2(a)=2s^2+2z^2, \text{ and } z^2=\frac{a-2s^2}{2} \text{ or } z=\sqrt{\frac{a-2s^2}{2}}.$$

$$\text{Hence } x=s+\sqrt{\frac{a-2s^2}{2}}, \text{ and } y=s-\sqrt{\frac{a-2s^2}{2}}.$$

II. $x^3=(s+z)^3=s^3+3s^2z+3sz^2+z^3$

$y^3=(s-z)^3=s^3-3s^2z+3sz^2-z^3$;

$$\therefore x^3+y^3(b)=2s^3+6sz^2; \text{ and } z^2=\frac{b-2s^3}{6s}, \text{ or } z=\sqrt{\frac{b-2s^3}{6s}}.$$

$$\text{Hence } x=s+\sqrt{\frac{b-2s^3}{6s}}, \text{ and } y=s-\sqrt{\frac{b-2s^3}{6s}}.$$

III. $x^4=(s+z)^4=s^4+4s^3z+6s^2z^2+4sz^3+z^4$,

$y^4=(s-z)^4=s^4-4s^3z+6s^2z^2-4sz^3+z^4$;

$\therefore x^4+y^4(c)=2s^4+12s^2z^2+2z^4$ is a quadratic equation from which the value of z may be found.

IV. $x^5=(s+z)^5=s^5+5s^4z+10s^3z^2+10s^2z^3+5sz^4+z^5$,

$y^5=(s-z)^5=s^5-5s^4z+10s^3z^2-10s^2z^3+5sz^4-z^5$;

$\therefore x^5+y^5(d)=2s^5+20s^3z^2+10sz^4$ is a quadratic equation from which the value of z may be found(b).

(b) In reviewing these operations, it may be observed, that those terms where the index of z is an *odd* number destroy each other in the successive

88. Let $x+y=2s$ and $x-y=2z$ as before, and let $\frac{x}{y}+\frac{y}{x}=a'$; $\frac{x^2}{y}+\frac{y^2}{x}=b'$; $\frac{x^3}{y}+\frac{y^3}{x}=c'$; and $\frac{x^4}{y}+\frac{y^4}{x}=d'$; then, by means of the equations in the preceding Article (87), the values of x and y may be found in terms of the known quantities, s, a', b', c', d'.

I. $\frac{x}{y}+\frac{y}{x}=a'$, $\therefore x^2+y^2=a'xy=a'(s+z)(s-z)=a'(s^2-z^2)$.

But by CASE I. (87.) $x^2+y^2=2s^2+2z^2$;

Hence $a's^2-a'z^2=2s^2+2z^2$,

and $z^2=\frac{(a'-2)s^2}{a'+2}$ or $z=\sqrt{\frac{(a'-2)s^2}{a'+2}}$;

$\therefore x=s+\sqrt{\frac{(a'-2)s^2}{a'+2}}$, and $y=s-\sqrt{\frac{(a'-2)s^2}{a'+2}}$.

II. $\frac{x^2}{y}+\frac{y^2}{x}=b'$, $\therefore x^3+y^3=b'xy=b'(s^2-z^2)$.

By CASE II. (87.) $x^3+y^3=2s^3+6sz^2$;

$\therefore b(s^2-z^2)=2s^3+6sz^2$,

and $z^2=\frac{(b'-2s)s^2}{b'+6s}$, or $z=\sqrt{\frac{(b'-2s)s^2}{b'+6s}}$.

Hence $x=s+\sqrt{\frac{(b'-2s)s^2}{b'+6s}}$, and $y=s-\sqrt{\frac{(b'-2s)s^2}{b'+6s}}$.

III. $\frac{x^3}{y}+\frac{y^3}{x}=c'$, $\therefore x^4+y^4=c'xy=c'(s^2-z^2)$.

By CASE III. (87.) $x^4+y^4=2s^4+12s^2z^2+2z^4$;

Hence $c'(s^2-z^2)=2s^4+12s^2z^2+2z^4$ is a quadratic equation, by which the value of z may be found.

---

series; hence, if the operations had been continued to x^6+y^6 and x^7+y^7, the resulting equations would have been equations of *six* dimensions in a *cubic* form; if they had been carried on to x^8+y^8 and x^9+y^9, the resulting equations would have been equations of *eight* dimensions in a *biquadratic* form. Hence the Problem of "Given the sum of two numbers, and the sum of their *n*th powers, to find the numbers themselves," may be solved as far as the 9th power, by means either of *quadratic*, *cubic*, or *biquadratic* equations.

IV. $\frac{x^4}{y}+\frac{y^4}{x}=d'$, $\therefore x^5+y^5=d'xy=d'(s^2-z)$.

By Case IV. (87). $x^5+y^5=2s^5+20s^3z+10sz^4$; and by equating these two values of x^5+y^5, there arises a quadratic equation by which the value of z may be determined.

---

89. Let $x+y=s$, and $xy=p$; then the sums of the several powers of x and y may be found in terms of the known quantities p and s, in the following manner.

I. $x^2+2xy+y^2=s^2$;

$$\therefore x^2+y^2=s^2-2xy=s^2-2p.$$

II. $(x^2+y^2)(x+y)=(s^2-2p)s$,

or $x^3+y^3+xy(x+y)=s^3-2ps$,

i.e. $x^3+y^3+ps=s^3-2ps$;

$$\therefore x^3+y^3=s^3-3ps.$$

III. $(x^3+y^3)(x+y)=(s^3-3ps)s$,

or $x^4+y^4+xy(x^2+y^2)=s^4-3ps^2$,

i.e. $x^4+y^4+p(s^2-2p)=s^4-3ps^2$;

$$\therefore x^4+y^4=s^4-4ps^2+2p^2.$$

IV. $(x^4+y^4)(x+y)=(s^4-4ps^2+2p^2)s$,

or $x^5+y^5+xy(x^3+y^3)=s^5-4ps^3+2p^2s$,

i.e. $x^5+y^5+p(s^3-3ps)=s^5-4ps^3+2p^2s$;

$$\therefore x^5+y^5=s^5-5ps^3+5p^2s.$$

or in general $x^n+y^n=s^n-nps^{n-2}+n\frac{(n-3)}{2}p^2s^{n-4}-$&c.

Example 1.

The sum of two numbers is 6, and the sum of their fifth powers is 1056. What are the numbers?

This Example belongs to Case IV. Art 87. where $s=3$, and $d=1056$.

The equation to find the value of z is

$$2s^5+20s^3z^2+10sz^4=d,$$

$$\text{or } 486+540z^2+30z^4=1056;$$

Divide by 6, $81+90z^2+5z^4=176$;

$$\therefore z^4+18z^2=19.$$

By **Rule I.** $z^4+18z^2+81=100$,

or $z^2+9=10$; $\therefore z^2=1$, and $z=1$.

Hence $x=s+z=3+1=4$,

$$y=s-z=3-1=2.$$

Ex. 2. There are two numbers whose sum is 18, and the square of the greater divided by the lesser *plus* the square of the lesser divided by the greater is 27; What are the numbers?

In Case II. (88.) $s=9$, and $b'=27$; hence $z=\sqrt{\frac{(b'-2s)s^2}{b'+6s}}=\sqrt{\frac{9\times 81}{27+54}}=\sqrt{\frac{9\times 81}{81}}=\sqrt{9}=3$; $\therefore x=s+z=9+3=12$, and $y=s-z=9-3=6$; and the two numbers are 12 and 6.

Ex. 3. The sum of two numbers is $5\,(s)$, and their product $6\,(p)$; What is the sum of their 4th powers?

By Case III. (Art. 89.) $x^4+y^4=s^4-4ps^2+2p^2=625-600+72=25+72=97$.

CHAP. VI.

ON RATIOS, PROPORTION, AND VARIATION

XXVII.

Definitions.

90. By **Ratio** is meant the relation which one quantity bears to another, with respect to magnitude. It is evident that this relation can exist only between quantities of a similar kind; thus, a *number* must be compared with a number; a *line* with a line; &c. &c.; and it would be absurd to compare a certain number of *feet* with a certain number of *pounds*; &c. &c.

•91. There are two ways in which the magnitude of quantities may be compared. In the first place, they may be compared with regard to their *difference;* and then the question asked, is, " How much one quantity is greater or less than another." The relation which quantities bear to each other in this respect, is called their *Arithmetical* Ratio. The other way in which they may be compared, is, by inquiring, " How often one quantity is contained in the other." This relation between quantities is called their *Geometrical* Ratio. The term *ratio*, when simply applied, is generally understood in the latter sense; and it is in this sense that the word will be made use of in the present Chapter.

92. In considering how often one quantity is contained in another, the natural process is to *divide* the one by the other. Thus, in comparing the number 12 with the numbers 4 and 3, we know that 4 is contained in 12 *three* times, and that 3 is contained in the same number *four* times; from which we infer that the ratio of 12 : 3(a) is *greater* than the ratio of 12 : 4, the *magnitude* of a ratio being measured by the *number of times* one quantity is contained in another. For the same reason, the ratio of 11 : 7 is said to be *less* than the ratio of 11 : 5. When the ratio is thus expressed, the first term of it is called the *antecedent*, the last term the *consequent*, of that ratio.

93. From this mode of estimating the magnitude of a ratio, it appears that when the consequent of a ratio is not an *aliquot part* of the antecedent, the value of the ratio must be expressed by a *fraction*, whose *numerator* is the antecedent, and *denominator* the consequent of that ratio. Thus the magnitude of the ratio of 15 : 7 is expressed by the fraction $\frac{15}{7}$, and of the ratio 4 : 13 by the fraction $\frac{4}{13}$. When the antecedent of a ratio is greater than the consequent, it is called a ratio of *greater inequality;* when the antecedent is less than the consequent, a ratio

---

(a) In expressing the ratio of two quantities, the word " *to* " is generally supplied by two dots; thus, the ratio of " *a* to *b* " is expressed by " *a* : *b*."

a ratio of *lesser inequality;* and if the two terms of a ratio be the *same,* then it is said to be a ratio of *equality.*

94. The foregoing definitions evidently apply only to those instances, in which the consequent of a ratio is contained a certain number of times in the antecedent, or to those in which the magnitude of the ratio may be expressed by some definite fraction. It does not therefore comprehend such ratios as $\sqrt{2} : 5$; $\sqrt{3} : \sqrt[3]{7}$; $4 : \sqrt[4]{3}$; &c.; where the values of the quantities $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$, $\sqrt[4]{3}$ &c. can only be expressed in decimal fractions which do not terminate. The ratio which exists between quantities of this latter kind, when the radical quantity is expressed by a decimal fraction, is called their *approximate* ratio.

95. *Proportion* consists in the *equality of ratios;* thus, since 4 is contained in 12, the same number of times that 6 is in 18, the ratio of $12 : 4$ is said to be equal to the ratio of $18 : 6$, or, in other words, that $12 : 4 :: 18 : 6$.[b] Of the four terms of which every proportion consists, the first and last terms are called the *extremes,* and the second and third the *means* of that proportion.

96. If there be a set of quantities related together in the following manner, viz. $a : b :: b : c :: c : d :: d : e$, &c. where the consequent of every preceding ratio is the antecedent of the following one, then the quantities a, b, c, d, e, &c. are said to be in *continued* proportion; and if only *three* quantities be concerned, as in the proportion $a : b :: b : c$, then b is said to be a *mean proportional* between the two extremes a and c.

97. Since the proportion $a : b :: c : d$ expresses the equality of the ratios $a : b$ and $c : d$; and since the magnitude of the ratio $a : b$ is measured by the fraction $\frac{a}{b}$, and that of the ratio

---

(b) In stating a proportion, the words "*is to*" and "*to*" are generally supplied by two dots, and the word "*as*" by four dots; thus, the proportion "*a* is to *b* as *c* to *d*," is expressed by "$a : b :: c : d$."

ratio $c:d$ by the fraction $\frac{c}{d}$, it follows that $\frac{a}{b}=\frac{c}{d}$, or that "when four quantities are proportional, the quotient of "the first divided by the second is equal to the quotient "of the third divided by the fourth:" and *vice versâ*, "if "there be four quantities a, b, c, d, such, that $\frac{a}{b}=\frac{c}{d}$, then "those four quantities are proportional, or $a:b::c:d$."

XXVIII.

On the Comparison and Composition of Ratios.

98. *On the comparison of Ratios.*

I. Since the ratio of $a:b$ may be expressed by the fraction $\frac{a}{b}$, let the numerator and denominator of this fraction be multiplied by any quantity m (m being either *integral* or *fractional*), then $\frac{ma}{mb}=\frac{a}{b}$, and $\therefore$ the ratio of $ma:mb$ is the same with the ratio of $a:b$; from which we infer, that "if "the terms of a ratio be multiplied or divided by the same "quantity, it does not alter the value of the ratio." From hence also it appears, that a ratio is reduced to its *lowest terms* by dividing its antecedent and consequent by their greatest common measure.

II. "Ratios are compared together by reducing the fractions by which their values are respectively represented, "to a common denominator." Thus, the ratio of $8:5$ is represented by the fraction $\frac{8}{5}$, and the ratio of $9:6$ by the fraction $\frac{9}{6}$; reduce these fractions to others of the same value, having a common denominator, and they become $\frac{48}{30}$ and $\frac{45}{30}$ respectively; and since $\frac{48}{30}$ is greater than $\frac{45}{30}$, the ratio $8:5$ is greater than the ratio of $9:6$.

III. "A ratio of greater inequality is *diminished*, and a "ratio of lesser inequality is *increased*, by *adding* the same

" quantity to both its terms." Let $a+b:a$ represent a ratio of *greater inequality*, and let x be added to each of its terms, and it becomes the ratio of $a+b+x:a+x$. Now the ratio of $a+b:a=\frac{a+b}{a}$, and that of $a+b+x:a+x=\frac{a+b+x}{a+x}$; let these fractions be reduced to others of same value having a common denominator, and they become $\frac{a^2+ab+ax+bx}{a(a+x)}$ and $\frac{a^2+ab+ax}{a(a+x)}$ respectively; and since $a^2+ab+ax+bx$ is evidently greater than $a^2+ab+ax$, the ratio of $a+b:a$ is greater than the ratio of $a+b+x:a+x$; i.e. the ratio of $a+b:a$ has been *diminished* by adding x to each of its terms. Next, let $a-b:a$ represent a ratio of *lesser inequality*; then proceeding with the fractions $\frac{a-b}{a}$ and $\frac{a-b+x}{a+x}$, as in the former instance, the resulting fractions are $\frac{a^2-ab+ax-bx}{a(a+x)}$ and $\frac{a^2-ab+ax}{a(a+x)}$; and since $a^2-ab+ax-bx$ is less than $a^2-ab+ax$, the ratio of $a-b:a$ is less than the ratio of $a-b+x:a+x$, and consequently the ratio of $a-b:a$ has been *increased* by adding x to each of its terms. In the same manner it might be shewn that " a ratio of greater inequality is *increased*, and " a ratio of lesser inequality is *diminished*, by *subtracting* " the same quantity from each of its terms."

99. *On the composition of Ratios.*

I. Ratios are compounded together by multiplying their antecedents together for a *new* antecedent, and their consequents together for a *new* consequent. Thus, if the ratio of $a:b$ be compounded with the ratio of $c:d$, the resulting ratio is that of $ac:bd$; or if the ratios $4:3$; $5:2$; and $7:1$, be compounded together, there results the ratio of $4\times5\times7:3\times2\times1$, or of $140:6$, or (dividing each term by 2) of $70:3$.

II. If the *same* ratio be compounded with itself *once*, *twice*, *thrice*, &c., the resulting ratios are those of $a^2:b$; $a^3:b^3$; $a^4:b^4$, &c. &c. The ratio of $a^2:b^2$ is called the *duplicate* ratio of $a:b$; a^3 [illegible] b^3 the *triplicate;* $a^4:b^4$ the *quadruplicate;* &c. &c.; and as these ratios receive their denominations from the *indices* of the several powers of a and b, the ratio of $\sqrt{a}:\sqrt{b}$ is called the *subduplicate* ratio of $a:b$; the ratio of $\sqrt[3]{a}:\sqrt[3]{b}$, the *subtriplicate;* &c. &c.

III. "If a set of ratios, whereof the consequent of the "preceding ratio is the same with the antecedent of the "succeeding one, be compounded together, the resulting "ratio is that of the *first antecedent* to the *last consequent*." Thus, when the ratios of $a:b$; $b:c$; $c:d$; $d:e$; are compounded together, the resulting ratio is that of $abcd:bcde$, or (dividing by bcd) that of $a:e$, or of the *first antecedent* : the *last consequent;* and the same will be the case whatever be the number of ratios.

IV. "A ratio of *greater inequality* compounded with "another ratio, *increases* it; and a ratio of *lesser inequality* "compounded with another ratio, *diminishes* it." Thus, let $1+n:1$ represent a ratio of greater inequality, and let it be compounded with the ratio $a:b$, the resulting ratio is that of $a+na:b$, which is evidently *greater* than the ratio of $a:b$; on the other hand, let $1-n:1$ represent a ratio of lesser inequality, and let it be compounded with the ratio of $a:b$, then the resulting ratio is that of $a-na:b$, which is evidently *less* than the ratio of $a:b$.

EXAMPLES.

Ex. 1. Reduce the ratio of $360:315$, and $1595:667$, to their lowest terms.

Ex. 2. Reduce the ratio of $a^3+2a^2x:a$ to its lowest terms.

Ex. 3. Which is the *greatest*, the ratio of $16:15$, or that of $17:14$?

Ex. 4. Which is the *least* of the three ratios, $20:17$, $22:18$,

$22:18$, or $25:23$? and which is the *greatest* of the three ratios, $8:7$; $6:5$; and $10:9$?

Ex. 5. Which is the *greatest*, the ratio of $a+2:\frac{1}{2}a+4$, or that of $a+4:\frac{1}{2}a+5$?

ANSWER, The ratio of $a+4:\frac{1}{2}a+5$.

Ex. 6. Compound together the ratios of $11:3$, $7:2$, and $5:9$. ANSW. $385:54$.

Ex. 7. Compound together the ratios of $15:12$, $6:7$, and $9:4$; and then reduce the resulting ratio to its *lowest terms*. ANSW. $135:56$.

Ex. 8. Express in the *simplest* terms the ratio compounded of $a^2-x^2:a^2$, $a+x:b$, and $b:a-x$.

ANSW. $(a+x)^2:a^2$.

Ex. 9. If the ratios of $x+y:a$, $x-y:b$, and $b:\frac{x^2-y^2}{a}$, be compounded together, shew that the resulting ratio is a ratio of *equality*.

Ex. 10. If the ratios of $3a+2:6a+1$, and of $2a+3:a+2$, be compounded together, is the resulting ratio a ratio of *greater* or *lesser* inequality?

ANSW. A ratio of *greater* inequality.

Ex. 11. What are the *least* numbers in the ratio compounded of the three following ratios, viz. the ratio of $7:5$. the *duplicate* ratio of $4:9$, and the *triplicate* ratio of $3:2$?

ANSW. 14 and 15.

Ex. 12. Compound the *subduplicate* ratio of $x^2:y^2$, with the *quadruplicate* ratio of $\sqrt{x}:\sqrt{y}$. ANSW. $x^3:y^3$.

XXIX.

On Proportion.

100. The most useful Theorems relating to proportional quantities are the following.

TH. 1. "If four quantities be proportional, the product of the extremes will be equal to the product of the means;

for let $a:b::c:d$, then, by Art. 97, $\frac{a}{b}=\frac{c}{d}$, $\therefore ad=bc$. From hence also it follows, "that if any three terms of a proportion be known, the fourth may be found;" for, from the equation $ad=bc$, we have $a=\frac{bc}{d}$; $a=\frac{ad}{c}$; $c=\frac{ad}{b}$ and $d=\frac{bc}{\ }$.

TH. 2. The converse of the foregoing Theorem is also true; viz. "If the product of any two quantities be equal to "the product of two others, those four quantities will con-"stitute a proportion, provided that the terms of one pro-"duct be made the *means*, and the terms of the other pro-"duct be made the *extremes*, of such proportion." Thus, if the four quantities a, b, c, d, be such that $ad=bc$, then (dividing by bd) $\frac{a}{b}=\frac{c}{d}$; $\therefore$ by Art. 97, $a:b::c:d$.

TH. 3. "If three quantities be proportional, the pro-"duct of the two extremes is equal to the square of the "mean;" for, if $a:b::b:c$, then by THEOR. 1, $ac=b^2$. From hence also it follows, that "a mean proportional between any two quantities is equal to the square root of their product;" for let x be a mean proportional between a and c, then $a:x::x:c$, $\therefore x^2=ac$, and $x=\sqrt{ac}$.

TH. 4. "If four quantities be proportional, they will also be proportional when taken *inversely* or *alternately*;" thus if $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$, *invert* the fractions, then $\frac{b}{\ }=\frac{d}{\ }$ $\therefore b:a::d:c$. Again, since $ad=bc$, then (dividing by cd, we have $\frac{ad}{cd}=\frac{bc}{cd}$, or $\frac{a}{c}=\frac{d}{d}$; $\therefore a:c::b:d$.

TH. 5. "If there be six proportional quantities, and the "first be to the second as the third to the fourth; and the "third to the fourth as the fifth to the sixth; then will the "first be to the second as the fifth to the sixth." For let

$a:b::c:d$, and $c:d::e:f$; then $\frac{a}{b}=\frac{c}{d}$; and $\frac{c}{d}=\frac{e}{f}$; $\therefore \frac{a}{b}=\frac{e}{f}$, or by Art. 97, $a:b::e:f$.

Th. 6. "If four quantities be proportional, then the *sum* " or *difference* of the first and second will be to the second as " the *sum* or *difference* of the third and fourth is to the fourth."

For let $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$; *add* or *subtract* 1 from each side of the equation; then $\frac{a}{b}\pm 1=\frac{c}{d}\pm 1$, $\therefore \frac{a\pm b}{b}=\frac{c\pm d}{d}$, consequently, by Art. 97, $a\pm b:b::c\pm d:d$.

Th. 7. "If four quantities be proportional, the *first* is to " the *sum* or *difference* of the first and second, as the *third* " to the *sum* or *difference* of the third and fourth." For by Theor. 6, $a\pm b : b :: c\pm d : d$, and alternately $a\pm b : c\pm d :: b:d$; but by Theor. 4, $b:d::a:c$; hence, by Theor. 5, $a\pm b : c\pm d :: a : c$, and alternately $a\pm b : a :: c\pm d : c$; $\therefore$ *inversely*, $a : a\pm b :: c : c\pm d$.

Th. 8. "If four quantities be proportional, then the *sum* " of the first and second is to their *difference*, as the *sum* of " the third and fourth is to *their* difference." For by Theor. 6, $\frac{a+b}{b}=\frac{c+d}{d}$, and $\frac{a-b}{b}=\frac{c-d}{d}$; *invert* the two last fractions, then $\frac{b}{a-b}=\frac{d}{c-d}$; hence $\frac{a+b}{b}\times\frac{b}{a-b}=\frac{c+d}{d}\times\frac{d}{c-d}$, or $\frac{a+b}{a-b}=\frac{c+d}{c-d}$; $\therefore$ by Art. 97, $a+b : a-b :: c+d : c-d$.

Th. 9. "If four quantities be proportional, and any *equi-* " *multiples* or *equal parts* whatever be taken of the first " and second, and also of the third and fourth; then will " the resulting quantities, taken in the same order, be still " proportional." For let $a : b :: c : d$; then, by Case I, Art. 98, the ratio of $ma : mb$ is the same with the ratio of $a : b$; and for the same reason, the ratio of $nc : nd$ is the same with the ratio of $c : d$; hence (Art. 95) $ma : mb :: nc : nd$, where m and n may be any quantities whatever, either *integral* or *fractional*.

TH. 10. The same theorem is true, "if any *equimultiples* "or *equal parts* whatever be taken of the *first and third*, "and also of the *second and fourth*;" for since $\frac{a}{b}=\frac{c}{d}$, multiply each side of the equation by $\frac{m}{n}$, then $\frac{ma}{nb}=\frac{mc}{nd}$. $\therefore ma:nb::mc:nd$, where m and n may be any quantities whatever, either *integral* or *fractional*.

TH. 11. "If four quantities be proportional, any *powers* or *roots* of those quantities will also be proportional." For since $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, $\therefore a^n:b^n::c^n:d^n$, where n may be any number either *integral* or *fractional*.

TH. 12. If the corresponding terms of two sets of pro-"portionals be multiplied together, or divided by each "other, the resulting quantities taken in order will still be "proportional." Thus, let

$$\left.\begin{array}{c} a:b::c:d \\ \text{and} \\ e:f::g:h \end{array}\right\} \left.\begin{array}{l} \text{then } \frac{a}{b}=\frac{c}{d} \\ \\ \text{and } \frac{e}{f}=\frac{g}{h} \end{array}\right\} \text{hence } \frac{ae}{bf}=\frac{cg}{dh} \text{ or } ae:bf::cg:dh.$$

Again, by TH. 1, $ad=bc$, and $eh=fg$; $\therefore \frac{ad}{eh}=\frac{bc}{fg}$; hence, by TH. 2, $\frac{a}{e}:\frac{b}{f}::\frac{c}{g}:\frac{d}{h}$. The same will evidently be true of any number of proportions.

TH. 13. "If there be two rows of proportional quantities, "whereof the *second* and *fourth* of the first row are the "same with the *first* and *third* of the second row, then will "the remaining quantities, taken in order, be proportional;" thus, let $a:b::c:d$,

and $b:e::d:f$; then, by THEOR. 12, $ab:be::cd:df$, or (reducing each ratio to its lowest terms) $a:e::c:f$.

TH. 14. "If there be a set of proportional quantities, "$a:b::c:d::e:f::g:h$ &c. &c., then will the *first* be "to the *second* as the *sum of all the antecedents* to the *sum* "*of all the consequents.*"

For, since $ab=ba$, and (by THEOR^s. 1 & 5) $ad=bc$, $af=be$, $ah=bg$, &c. we have $ab+ad+af+ah+$&c.$=ba+bc+be+bg+$&c., or $a(b+d+f+h+\&c.)=b(a+c+e+g+\&c.)$ $\therefore$ (by THEOR. 2.) $a:b::a+c+e+g+\&c.:b+d+f+h+\&c.$

TH. 15. "If $a:b::b:c::c:d::d:e$ &c. as in Art. 96.
"then $a:c::a^2:b^2$, or in the *duplicate* ratio of $a:b$;
"$a:d::a^3:b^3$, or in the *triplicate* ratio of $a:b$;
"$a:e::a^4:b^4$, or in the *quadruplicate* ratio of $a:b$;"
&c. &c. &c. &c.

For $a:b::a:b$;
and $b:c::a:b$;
$\therefore$ by THEOR. 12, $a:c::a^2:b^2$.

Again, $a:c::a^2:b^2$;
but $c:d::a:b$;
$\therefore$ by THEOR. 12, $a:d::a^3:b^3$;

Moreover, $a:d::a^3:b^3$;
but $d:e::a:b$;
$\therefore$ by THEOR. 12, $a:e::a^4:b^4$.

&c. &c. &c. &c.

101. The following Examples are intended to illustrate the use of the foregoing Theorems.

EXAMPLE I.

To divide the number 60 into two such parts, that the *product* shall be to the *sum of the squares* :: 2 : 5.

Let x=one part;
then $60-x$=the other part,
$(60-x)\times x=60x-x^2$=the *product*,
and $x^2+(60-x)^2=2x^2+3600-120x$=*sum of the squares*.
Hence, by the question, $60x-x^2:2x^2+3600-120x::2:5$;
$\therefore$ by THEOR. 1, $(60x-x^2)\times 5=(2x^2+3600-120x)\times 2$,
or $300x-5x^2=4x^2+7200-240x$;
by *transposition & division*, $x^2-60x=-800$;
$\therefore x^2-60x+900=900-800=100$,
and $x-30=\pm 10$;
or $x=30\pm 10=40$ *or* 20, the [parts required.

Ex. 2.

The number 20 is divided into two parts, which are to each other in the *duplicate* ratio of 3 : 1. Find a *mean proportional* between those parts.

Let x=*greater* part,
then $20-x$=*lesser* part;
$\therefore$ by the question, $x:20-x::3^2:1^2::9:1$.
Hence, by THEOR. 1, $x=180-9x$,
or $10x=180$;
$\therefore x=18$=*greater* part,
and $20-x=20-18=2$=*lesser* part.

By THEOR. 3, a *mean proportional* between 18 and 2 is equal to $\sqrt{18\times 2}=\sqrt{36}=6$, the number required.

Ex. 3.

If $(a+x)^2:(a-x)^2::x+y:x-y$, shew that $a:x::\sqrt{2a-y}:\sqrt{y}$.
By *expansion*, $a^2+2ax+x^2:a^2-2ax+x^2::x+y:x-y$.
By THEOR. 8, $2a^2+2x^2:4ax::2x:2y$.
Divide by 2, then $a^2+x^2:2ax::x:y$;
$\therefore$ by THEOR. 1, $(a^2+x^2)\times y=2ax\times x=2a\times x^2$.
Hence, by THEOR. 2, $a^2+x^2:x^2::2a:y$.
By THEOR. 6, $a^2:x^2::2a-y:y$;
and by THEOR. 11, (n being $\frac{1}{2}$) $a:x::\sqrt{2a-y}:\sqrt{y}$.

Ex. 4.

Ex. 4.

If $x : y$ in the *triplicate* ratio of $a : b$, and $a : b :: \sqrt[3]{c+x} : \sqrt[3]{d+y}$, shew that $dx=cy$.

Since $x : y :: a^3 : b^3$,
and by Theor. 11, $a^3 : b^3 :: c+x : d+y$;
∴ by Theor. 5, $x : y :: c+x : d+y$,
or $c+x : d+y :: x : y$,
and by Theor. 4, $c+x : x :: d+y : y$;
∴ by Theor. 6, $c : x :: d : y$;
and by Theor. 1, $dx=cy$.

Ex. 5.

There are two numbers whose *product* is 24, and the *difference of their cubes* : *cube of their difference* :: 19 : 1. What are the numbers?

Let $x=$ *greater* number,
and $y=$ *lesser* number.

Then, by the question, $xy=24$,
and $x^3-y^3 : (x-y)^3 :: 19 : 1$.

By *expansion*, $x^3-y^3 : x^3-3x^2y+3xy^2-y^3$ or $(x-y)^3 :: 19 : 1$.
By Theor. 6, $3x^2y-3xy^2 : (x-y)^3 :: 18 : 1$,
or $3xy\times(x-y) : (x-y)^3 :: 18 : 1$.
Divide by $x-y$, then $3xy : (x-y)^2 :: 18 : 1$;
but $xy=24$; ∴ $72 : (x-y)^2 :: 18 : 1$.

Hence, by Theor. 1, $18\times(x-y)^2=72$,
or $(x-y)^2=4$;
∴ $x-y=2$.

Again, $x^2-2xy+y^2=4$,
and $4xy=96$.

∴ $x^2+2xy+y^2=100$,

$$\left.\begin{array}{l} \text{or } x+y=10, \\ \text{but } x-y=2; \end{array}\right\} \begin{array}{l} \therefore x=\frac{1\text{[illegible]}}{2}=6. \\ y=\frac{8}{2}=4. \end{array}$$

Ex. 6. To divide the number 24 into two such parts, that their *product* shall be to the *sum of their squares* :: 3 : 10. ANSWER, 18 and 6.

Ex. 7. There are two numbers which are to each other as 3 : 2. If 6 be *added to the greater*, and *subtracted from the lesser*, the *sum* will be to the *remainder* :: 3 : 1. What are the numbers? ANSW. 24 and 16.

Ex. 8. There are two numbers which are to each other in the *duplicate* ratio of 4 : 3, and 24 is a *mean proportional* between them. What are the numbers? ANSW. 32 and 18.

Ex. 9. If $^{a^2-x^2}=4a$; shew that $a+x : 2a :: 2b : a-x$.

Ex. 10. If $x^2 : y^2 :: 36 : 25$, and $2x+y : x+2$ in a ratio compounded of the ratios of 17 : 2 and 2 : 7; What are the numbers? ANSW. 12 and 10.

Ex. 11. There are two numbers whose product is 135, and the *difference of their squares* is to the *square of their difference* :: 4 : 1. What are the numbers? ANSW. 15 and 9.

XXX.

On Variation.

102. If the quantities under consideration be of a *variable* nature, then their relation to each other may be expressed in the following manner.

I. Let A and B be two variable quantities so related to each other, that whilst the value of A is changed to a, the value of B is changed to b; then, if these two quantities A and B always bear the same ratio to each other, i. e. if $A : B :: a : b$ (or, by Theor. 4. of Proportion, $A : a :: B : b$) throughout the whole period of their variation, they are said to vary *directly* as each other.

EXAM. Suppose a body to move uniformly along, at the rate of 3 feet in one second of time; then in the *first* second it would describe 3 feet, in *two* seconds 6 feet, in *three* seconds 9 feet, &c. &c.; hence, whilst the time varies through 1, 2, 3, 4, &c. seconds, the space varies through 3, 6, 9, 12, &c. feet; but the numbers 3, 6, 9, &c. are respectively in the same ratio with the numbers 1, 2, 3, &c. When a body moves uniformly, therefore, "the space varies *directly* as the time."

II. If the relation between A and B be such, that whilst A by *increasing* is changed to a, and B by *decreasing* is changed to b, in such manner, that $A : a :: \frac{1}{B} : \frac{1}{b}$ (or) $:: b : B$ throughout the whole period of their variation, then A is said to vary *inversely* as B.

EXAM. The area of a triangle is equal to half the rectangle contained by its base and perpendicular altitude; if, therefore, the *form* of the triangle be changed whilst its *area* remains the same, it is evident that as its *altitude* increases its *base* must decrease. Let A and B represent its altitude and base at one period of its variation, and a and b its altitude and base at any other period, then $\frac{A \times B}{2} = \frac{a \times b}{2}$ or $A \times B = a \times b$, $\therefore$ (by TH. 2. Prop^{n}.) $A : a :: b : B :: \frac{1}{B} : \frac{1}{b}$, i.e. "the altitude of a triangle whose area is given varies *inversely* as its base, and vice versâ."

III. If there be three variable quantities A, B, C, whose relation to each other is such, that whilst B is changed to b, and C to c, A is changed in the *compound* ratio of the change of B and C; i.e. if $A : a$ in the ratio compounded of the ratios of $B : b$ and $C : c$, or (Art. 99, I.), $A : a :: BC : bc$, then A is said to vary as B and C *conjointly*.

EXAM. Let A represent the *area*, B the *base*, and C the *perpendicular altitude* of a triangle; and when these are changed, let a represent the *area*, b the *base*, and c the *altitude* at any period of their variation; then $A=\frac{BC}{2}$ and $a=\frac{bc}{2}$, $\therefore A:a::\frac{BC}{2}:\frac{bc}{2}::BC:bc$, or "the area of a "triangle varies as its *base and perpendicular altitude* "*conjointly.*"

IV. If the relation between the three quantities A, B, C be such, that when A is changed to a, B to b, and C to c, $B:b$ in the ratio compounded of the ratios of $A:a$ and $\frac{1}{C}:\frac{1}{c}$, or (Art. 99. I.) $B:b::\frac{A}{C}:\frac{a}{c}$, then B is said to vary *directly* as A, and *inversely* as C.

Exam. Let A, B, C, a, b, c represent the same quantities as in the last Example, then since $A=\frac{BC}{2}$, $B=\frac{2A}{C}$; and since $a=\frac{bc}{2}$, $b=\frac{2a}{c}$. Hence $B:b::\frac{2A}{C}:\frac{2a}{c}::\frac{A}{C}:\frac{a}{c}$, i.e. "the base will vary as the area *directly*, and as the perpendicular altitude *inversely*."

103. These several relations of variable quantities are often more briefly expressed by placing the mark $\propto$ between them; thus,

$A:a::B:b$, or "A varies as B," is expressed by $A\propto B$.
$A:a::\frac{1}{B}:\frac{1}{b}$, or "$A$ varies *inversely* as B," by . . $A\propto\frac{1}{B}$.
$A:a::BC:bc$, or "A varies as B & C *conjointly*," by $A\propto BC$.
$B:b::\frac{A}{C}:\frac{a}{c}$ $\left\{\begin{array}{l}\text{or "}B\text{ varies } \textit{directly} \text{ as } A,\\ \text{and } \textit{inversely} \text{ as } C\text{" . . .}\end{array}\right\}$ by . . $B\propto\frac{A}{C}$.

This notation is made use of in the following Theorems.

Th. 1. "If one quantity varies as another, it will also vary "as any *multiple* or *part* of the other, and any power or root "of the former will vary as the *same* power or root of the "latter." Thus, let $A\propto B$, then $A:a::B:b$; multiply the terms of last ratio by m, then (Art. 98, I.) $A:a::mB:mb$,

∴ (Art.

$\therefore$ (Art. 102, I.) $A\propto mB$, where m may be any number either *integral* or *fractional*. Again, since $A:a::B:b$, (by Th. 11 of Proportion,) $A^n:a^n::B^n:b^n$; $\therefore A^n\propto B^n$, where n may be any number whatever, *integral* or *fractional*.

Th. 2. "If one quantity varies as another, and each of "them be multiplied or divided by any quantity variable or "invariable, then will the products or quotients, thus arising, "vary as each other." Thus, let $A\propto B$, then $A:a::B:b$; let m be an *invariable* quantity, and multiply all the terms of the proportion by it, then $mA:ma::mB:mb$, $\therefore mA\propto mB$. Let C be a *variable* quantity, then we have

$$\left.\begin{array}{c} A:a::B:b \\ \text{and} \\ C:c::C:c \end{array}\right\} \therefore \text{by Th. 12 of Prop}^{\text{n}}. \left\{\begin{array}{l} AC:ac::BC:bc, \text{ or } AC\propto BC; \\ \text{and} \\ \frac{A}{C}:\frac{a}{c}::\frac{B}{C}:\frac{b}{c}, \text{ or } \frac{A}{C}\propto\frac{B}{C}. \end{array}\right.$$

Cor. 1. From hence it follows, that "if one quantity "varies as two others jointly, then either of those quanti- "ties varies as the first *directly* and the other *inversely*." Thus let $A\propto BC$, then, dividing each by C, $B\propto\frac{A}{C}$ or "as A directly and C inversely;" divide by B, then $C\propto\frac{A}{B}$ or "as A directly and B inversely."

Cor. 2. If the product of two quantities be invariable, then those quantities vary *inversely* as each other." For let $A\times B=m$, then $A=\frac{m}{B}$ which varies as $\frac{\ }{B}$, and $B=\frac{m}{A}$ which varies as $\frac{1}{A}$, m being a constant quantity.

Th. 3. "If one quantity varies as a second, and the "second as a third, then will the first quantity vary as the "third." For let $A\propto B$, then $A:a::B:b$; and let $B\propto C$, then $B:b::C:c$; $\therefore$ by Th. 5. of Propⁿ. $A:a::C:c$; hence $A\propto C$.

Th. 4. "If any two quantities vary as a third, then will "their *sum* or *difference* or the *square root of their product* "vary as the third." Thus let $A \propto C$ and $B \propto C$, then, by Th. 3, $A \propto B$; $\therefore A : a :: B : b$ or $A : B :: a : b$; and, by Th. 6 of Proportion, $A \pm B : B :: a \pm b : b$ or $A \pm B : a \pm b :: B : b$; but since $B \propto C$, $B : b :: C : c$, hence $A \pm B : a \pm b :: C : c$, or $A \pm B \propto C$.

Again, since

$A : a :: C : c$ } by Th. 12 of Prop^n. $AB : ab :: C^2 : c^2$,
and $B : b :: C : c$ } and, Th. 11 of Prop^n. $\sqrt{AB} : \sqrt{ab} :: C : c$.
Hence $\sqrt{AB} \propto C$.

Th. 5. "If the *square of the sum* of two quantities varies "as the *square of their difference*, then the *sum of their* "*squares* varies as their *product*." For let $(A+B)^2 \propto (A-B)^2$,

then $(A+B)^2 : (a+b)^2 :: (A-B)^2 : (a-b)^2$,
or $(A+B)^2 : (A-B)^2 :: (a+b)^2 : (a-b)^2$.

By Expansion, and } $2A^2 + 2B^2 : 4AB :: 2a^2 + 2b^2 : 4ab$,
by **Th.** 8 of Prop^n. } or $A^2 + B^2 : 2AB :: a^2 + b^2 : 2ab$;
$\therefore A^2 + B^2 : a^2 + b^2 :: 2AB : 2ab :: AB : ab$.
Hence $A^2 + B^2 \propto AB$.

Th. 6. "If there be two sets of quantities, A, B, C, D, "&c. and P, Q, R, S, &c. which vary as each other respectively, viz. $A \propto P$, $B \propto Q$, &c., then will the *products* "of those quantities vary as each other." For, let a, b, c, &c. p, q, r, &c. be corresponding values of A, B, C, &c. P, Q, R, &c. then, since $A \propto P$, $A : a :: P : p$

. . . $B \propto Q$, $B : b :: Q : q$
. . . $C \propto R$, $C : c :: R : r$
&c. &c.

$\therefore$ By **Theor.** 12 of Proportion,
ABC &c. : abc &c. :: PQR &c. : pqr &c.
Hence ABC &c. $\propto PQR$ &c.

Th. 7. "If any quantity A depends upon a set of quantities, P, Q, R, S, in such a manner, that if Q, R, S are con-

"stant, $A \propto P$; if P, R, S are constant, $A \propto Q$; &c. &c. "then if they *all* vary, A will vary as their *product.*"

For let A be changed

to x, by the variation of P to p, the rest being *constant,*
from x to y . . . . of Q to q . . . .
from y to z . . . . of R to r . . . .
from z to a . . . . of S to s . . . .

then, when *all* vary, we have

$$\left.\begin{array}{l} A : x :: P : p \\ x : y :: Q : q \\ y : z :: R : r \\ z : a :: S : s \end{array}\right\}$$

Hence, by composition of ratios, $A : a :: P\,Q\,R\,S : pqrs$ or $A \propto PQRS$; and the Theorem would evidently be true, whatever be the number of quantities P, Q, R, S, &c.

Th. 8. "If one quantity varies as another, it is *equal* to "that quantity multiplied into some *constant* quantity; and "the value of this constant quantity will be known, if the "actual relation between the two variable quantities at "some given period of their increase or decrease be "known." For let $A \propto B$, then $A : a :: B : b$, or $A : B :: a : b$, i. e. the ratio of $A : B$ is always the *same* through the whole period of their variation; let this ratio be that of $m : 1$, then $A : B :: m : 1$, and $A = mB$, or $m = \frac{A}{B}$. If therefore the corresponding values of A and B at any period of their variation be known, the value of m will be known.

Exam. The *space* described by a body descending perpendicularly near the surface of the Earth varies as the square of the *time;* let the *space*$=S$, the corresponding *time*$=T$, then by this Theorem $S = mT^2$; now it is known by experiment, that a body falls through a space of about 16 feet in the *first* second of its fall; hence, when $S = 16$, $T = 1$, $\therefore m = 16$, and the general relation between the space and time of a body thus falling is $S = 16T^2$.

Cor. Since $\frac{A}{B} = m$, it follows, "that if one quantity "varies as another, the fraction arising from dividing the "one quantity by the other, is a *constant* quantity."

CHAP. VII.

ON ARITHMETICAL AND GEOMETRICAL PROGRESSION.

XXXI.

DEFINITIONS.

104. If a series of quantities increase or decrease by the continual *addition* or *subtraction* of the same quantity, then those quantities are said to be in *Arithmetical* Progression. Thus the numbers 1, 2, 3, 4, 5, 6, &c. (which *increase* by the *addition* of 1 to each successive term), and the numbers 21, 19, 17, 15, 13, 11, &c. (which *decrease* by the *subtraction* of 2 from each successive term), are in arithmetical progression.

105. In general, if a represents the *first* term of any arithmetical progression, and b the *common difference*, then may the series itself be expressed by a, $a+b$, $a+2b$, $a+3b$, $a+4b$, &c., which will evidently be an *increasing* or a *decreasing* one, according as b is *positive* or *negative*. In the foregoing series, the *coefficient* of b in the *second* term is *one*; in the *third* term is *two*; in the *fourth* is *three*, &c.; i.e. the coefficient of b in any term is always *less by unity* than the number which denotes *the place of that term in the series*. Hence, if the number of terms in the series be denoted by (n), the nth or *last* term in the progression will be $a+(n-1)b$.

106. If a series of quantities increase or decrease by continual *multiplication* or *division* by the same quantity, then those quantities are said to be in *Geometrical* Progression. Thus the numbers 1, 2, 4, 8, 16, &c. (which *increase* by continual *multiplication* by 2), and the numbers 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. (which *decrease* by continual *division* by 3, or *multiplication* by $\frac{1}{3}$), are in Geometrical Progression.

107. In general, if a represents the *first term* of such a series, and r the *common multiple* or *ratio*, then may the series itself be represented by a, ar, ar^2, ar^3, ar^4, &c. which will evidently be an *increasing* or *decreasing* series, according as r is a *whole number* or a *proper fraction*. In the foregoing series, the *index of* r in any term is *less by unity* than the number which denotes *the place of that term in the series*. Hence, if the number of terms in the series be denoted by (n), the *last term* will be ar^{n-1}.

XXXII.

On Arithmetical Progression.

108. Let S be the *sum* of the series $a, a+b, a+2b, a+3b$, &c.; then

$$a+(a+b)\quad+(a+2b),\quad \&c.\ldots+\big(a+(n-2)b\big)+\big(a+(n-1)b\big)=S$$

$$\text{and } \big(a+(n-1)b\big)+\big(a+(n-2)b\big)+\big(a+(n-3)b,\big)\&c.\ldots+(a+b)\quad+a\quad=S$$

where the *lower* series is the same as the *upper* one, except that the order of the terms is inverted.

Add the two series together, and we have,

$$\big(2a+(n-1)b\big)+\big(2a+(n-1)b\big)+\big(2a+(n-1)b\big)+\&c. \text{ to } n \text{ terms}=2S,$$

$$\text{or } \big(2a+(n-1)b\big)n=2S;\ \therefore S=\big(2a+(n-1)b\big)\frac{n}{2}.\,^{(a)}$$

109. From the equation $\big(2a+(n-1)b\big)\cdot n=2S$, it appears, that if any three of the four quantities a, b, n, S are given, the fourth may be found. For we have,

i. By Art. 108 . . . . $S=\big(2a+(n-1)b\big)\frac{n}{2}$.

ii. By actual multiplication, $2an+bn^2-bn=2S$;

$$\therefore 2an=2S-bn^2+bn,$$

$$\text{and } a=\frac{2S-bn^2+bn}{2n}.$$

(a) Since the sum of any two terms $=\big(a+a+(n-1)b\big)=$ sum of *first* and *last* term, and since $S=\big(2a+(n-1)b\big)\frac{n}{2}$, it appears that the sum of the series is equal to *the sum of the first and last terms* (*or of any two terms equally distant from the first and last terms*), *multiplied into half the number of terms*.

III. . . . . Again, $bn^2 - bn = 2S - 2an$,

$$\text{or } (n^2-n)b = 2S - 2an;$$

$$\therefore b = \frac{2S-2an}{n^2-n}.$$

IV. To find n, we have, by transposition, $\Big\}$ $bn^2 + 2an - bn = 2S$,

$$\text{or } bn^2 + (2a-b)n = 2S;$$

$$\therefore n^2 + \frac{2a-b}{b}.n = \frac{2S}{b}.$$

Solve this *quadratic* equation, and it gives the value of n.

EXAMPLE 1.

Find the sum of the series 1, 3, 5, 7, 9, 11, &c. continued to 120 terms.

$$\left.\begin{array}{r} \text{Here } a=1, \\ b=2, \\ n=120; \end{array}\right\} \begin{array}{l} \therefore S=\left(2a+(n-1)b\right)\frac{n}{2}. \\ =\left(2\times 1+(120-1)2\right)\frac{120}{2}. \\ =\left(2+119\times 2\right)60=240\times 60=14400. \end{array}$$

Ex. 2.

Find the sum of the series 15, 11, 7, 3, -1, -5, &c. to 20 terms.

$$\left.\begin{array}{r} \text{Here } a=15, \\ b=-4, \\ n=20; \end{array}\right\} \begin{array}{l} \therefore S=\left(2a+(n-1)b\right)\frac{n}{2}. \\ =\left(2\times 15+(20-1)\times -4\right)\frac{20}{2}. \\ =(30-19\times 4)\times 10. \\ =(30-76)\times 10=-46\times 10=-460. \end{array}$$

Ex. 3.

Find the sum of 150 terms of the series $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, $\frac{7}{3}$, &c.

$$\left.\begin{array}{r} \text{Here } a=\frac{1}{3}, \\ b=\frac{1}{3}, \\ n=150; \end{array}\right\} \begin{array}{l} \therefore S=\left(2a+(n-1)b\right)\frac{n}{2}. \\ =\left(2\times\frac{1}{3}+(150-1)\times\frac{1}{3}\right)\frac{150}{2}. \\ =\left(\frac{2}{3}+\frac{149}{3}\right)75=\frac{151}{3}\times 75=3775. \end{array}$$

Ex. 4.

The *sum* of an arithmetic series is 1240, *common difference* -4, and *number of terms* 20. What is the *first term?*

$$\text{Here }\left.\begin{array}{l} S=1240,\\ b=-4,\\ n=20;\end{array}\right\}\begin{array}{l}\therefore a=\dfrac{2S-bn^2+bn}{2n}.\\ =\dfrac{2480+1600-80}{40}=\dfrac{4000}{40}=100.\end{array}$$

Hence the series is 100, 96, 92, 88, &c.

Ex. 5.

The *sum* of an arithmetic series is 1455, the *first term* 5, and the *number of terms* 30. What is the *common difference?*

$$\text{Here }\left.\begin{array}{l} S=1455,\\ a=5,\\ n=30;\end{array}\right\}\begin{array}{l}\therefore b=\dfrac{2S-2an}{n^2-n}.\\ =\dfrac{2910-300}{900-30}=\dfrac{2610}{870}=3.\end{array}$$

Hence the series is 5, 8, 11, 14, &c.

Ex. 6.

The *sum* of an arithmetic series is 567, the *first term* 7, the *common difference* 2. What are the *number of terms?*

$$\text{Here }\left.\begin{array}{l} S=567,\\ a=7,\\ b=2,\end{array}\right\}\begin{array}{ll}\therefore n^2+\dfrac{2a-b}{b}.n=\dfrac{2S}{b} & \\ \text{is } n^2+6n & =567;\\ \text{and } n^2+6n+9=567+9=576; & \\ \therefore n+3 & =24, \text{ or } n=21.\end{array}$$

Ex. 7.

How much ground does a person pass over in gathering up 200 stones placed in a straight line, at intervals of 2 feet from each other; supposing that he brings each stone *singly* to a basket standing at the distance of 20 yards from the first stone, and that he starts from the spot where the basket stands?

It is evident that the space passed over by this person will be *twice* the sum of an arithmetic series, whose *first term* is 20 *yards* (i. e. 60 feet), *common difference* 2 feet, and *number of terms* 200.

Here

$$\left.\begin{array}{r}\text{Here } a=60,\\ b=2,\\ n=200;\end{array}\right\}\begin{array}{l}\therefore S=\big(2a+(n-1)b\big)\frac{n}{2}.\\ \quad=\big(120+398\big)100.\\ \quad=518\times100=51800 \text{ feet.}\end{array}$$

Hence the distance required $=\overset{\text{feet,}}{103600}=\overset{\text{miles,}}{19}\;.\;\overset{\text{furlongs,}}{4}\;.\;\overset{\text{feet,}}{640}.$

Ex. 8.

A traveller bound to a place at the distance of 198 miles, goes 30 miles the *first* day, 28 the *second*, 26 the *third*, and so on. In how many days will he arrive at his journey's end?

$$\left.\begin{array}{r}\text{Here is given } a=30,\\ b=-2,\\ S=198,\end{array}\right\}\text{ to find the \textit{number of terms}.}$$

$$\text{Now } n^2+\frac{2a-b}{b}.n=\frac{2S}{b};\ \therefore n^2-31n=-\frac{2\times198}{2}=-198,$$

$$\text{and } n^2-31n+\frac{961}{4}=-198+\frac{961}{1}=\frac{169}{4}.$$

$$\text{Hence } n-\frac{31}{2}=\pm\frac{13}{2},\text{ and } n=\frac{31\pm13}{2}=22 \text{ or } 9.$$

To explain the apparent difficulty arising from the two positive values of n, which give us *two different periods* of the traveller's arrival at his journey's end, we must observe, that if the proposed series, 30, 28, 26, &c. be carried to 22 terms, the 16th term will be *nothing*, and the remaining six *negative;* by which is indicated the *rest* of the traveller on the 16th day, and his *return in the opposite direction* during the six days following; and this will bring him *again*, at the end of the 22d day, to the same point at which he was at the end of the 9th, viz. 198 miles from the place whence he set out.

Ex. 9.

There are a certain number of quantities in arithmetic progression, whose *common difference* is 2, and whose *sum* is equal to eight times their *number;* moreover, if 13 be added to the *second* term, and this sum be divided by the *number of terms*, the quotient will be equal to the *first term*. What are the numbers?

$$\left.\begin{array}{l}\text{Let the } \textit{first term}=x,\\ \text{and the } \textit{number of terms}=y,\end{array}\right\}\begin{array}{l}\text{then the } \textit{second} \text{ term will be } x+2,\\ \ldots\ \text{the } \textit{last} \text{ term } \ldots\ldots\ x+(y-1)\times 2.\end{array}$$

In the expression $\left(2a+(n-1)b\right)\frac{n}{2}$, substitute x for a, 2 for b, and y for n, and it becomes $\left(2x+(y-1)2\right)\frac{y}{2}\left(=xy+y^2-y\right)$, for the *sum* of the series.

By the question, $xy+y^2-y=8y$, or $y=9-x$.

$$\text{and } \frac{x+2+13}{y}=x.$$

$$\text{Hence, } \frac{x+2+13}{9-x}=x, \text{ or } x^2-8x=-15,$$

$$\therefore\ x^2-8x+16=16-15=1,$$

$$\text{and } x-4=\pm 1\ :\ \therefore\ x=5 \text{ or } 3,$$

$$y=9-x\qquad =4 \text{ or } 6.$$

From which it appears, that there are *two* sets of numbers which will answer the conditions required; viz. 5, 7, 9, 11, or 3, 5, 7, 9, 11, 13.

Ex. 10. Find the sum of 25 terms of the series,
2, 5, 8, 11, 14, &c. ANSWER, 950.

Ex. 11. Find the sum of 36 terms of the series,
40, 38, 36, 34, &c. ANSW. 180.

Ex. 12. Find the sum of 32 terms of the series,
1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, &c. ANSW. 280.

Ex. 13. The *sum* of an arithmetic series is 950, the *common difference* 3, and *number of terms* 25. What is the first term? ANSW. 2.

Ex. 14. The *sum* of an arithmetic series is 165, the *first term* 3, and the *number of terms* 10. What is the *common difference*? ANSW. 3.

Ex. 15. The *sum* of an arithmetic series is 440, *first term* 3, and *common difference* 2. What is the *number of terms?* Answ. 20.

Ex. 16 The *sum* of an arithmetic series is 54, *first term* 14, and *common difference* −2. What is the *number of terms?* Answ. 9, or 6.

Ex. 17. A person bought 47 sheep, and gave 1 shilling for the *first* sheep, 3 for the *second*, 5 for the *third*, and so on. What did *all* the sheep cost him? Answ. £. 110. 9*s*.

Ex. 18. A person began the year by giving away a *farthing* the *first* day, a *halfpenny* the *second*, *three farthings* the *third*, and so on. What money had he disposed of in charity at the *end of the year?* Answ. £.69.11*s*.6$\frac{3}{4}$*d*.

Ex. 19. *A* travels *uniformly* at the rate of 6 miles an hour, and sets off upon his journey 3 hours and 20 minutes before *B*; *B* follows him at the rate of 5 miles the *first* hour, 6 the *second*, 7 the *third*, and so on. In how many hours will *B* overtake *A?* Answ. In 8 hours.

Ex. 20. There are a certain number of quantities in arithmetic progression, whose *first term* is 2, and whose *sum* is equal to 8 times their number; if 7 be added to the *third* term, and that sum be divided by the number of terms, the quotient will be equal to the *common difference*. What are the numbers? Answ. 2, 5, 8, 11, 14.

XXXIII.

On Geometrical Progression.

110. Let S be the sum of the series a, ar, ar^2, ar^3, &c. (Art. 107), then

$$a+ar+ar^2+ar^3+\&c.\ldots ar^{n-2}+ar^{n-1}\qquad=S.$$

Multiply the equation by r, and it becomes

$$ar+ar^2+ar^3+\&c.\ldots ar^{n-2}+ar^{n-1}+ar^n=rS.$$

Subtract the *upper* equation from the *lower*, and we have,

$$ar^n-a=rS-S, \text{ or } (r-1)S=ar^n-a;$$

$$\text{and therefore, } S=\frac{ar^n-a}{r-1}.$$

If r is a *proper fraction*, then r and *its powers* are less than 1.

For the convenience of calculation, therefore, it is better in this case to transform the equation into $S=\frac{a-ar^n}{1-r}$, by multiplying the numerator and denominator of the fraction $\frac{ar^n-a}{r-1}$ by -1.

111. If l be the last term of a series of this kind, then $l=ar^{n-1}$, $\therefore rl=ar^n$; hence $S=\left(\frac{ar^n-a}{r-1}\right)=\frac{rl-a}{r-1}$. From this equation, therefore, if any three of the four quantities S, a, r, l, be given, the fourth may be found. For $S=\frac{rl-a}{r-1}$; $a=rl-(r-1)S$; $r=\frac{S-a}{S-l}$, and $l=\frac{(r-1)S+a}{r}$. The value of n cannot be found from the equation $S=\frac{ar^n-a}{r-1}$ except by means of *Logarithms*, as will be shewn in a future chapter.

EXAMPLE 1.

Find the sum of the series 1, 3, 9, 27, &c. to 12 terms.

$$\left.\begin{array}{l}\text{Here } a=1\\ r=3\\ n=12\end{array}\right\}\therefore S=\frac{ar^n-a}{r-1}=\frac{1\times 3^{12}-1}{3-1}.$$

$$=\frac{81^3-1}{2}.$$

$$=\frac{531441-1}{2}=\frac{531440}{2}=265720.$$

Ex. 2.

Find the sum of ten terms of the series $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}$, &c.

$$\left.\begin{array}{l}\text{Here } a=1\\ r=\frac{2}{3}\\ n=10\end{array}\right\}\therefore S=\frac{a-ar^n}{1-r}=\frac{1-\left(\frac{2}{3}\right)^{10}}{1-\frac{2}{3}}=\frac{\left(1-\left(\frac{2}{3}\right)^{10}\right)3}{3-2}=\left(1-\left(\frac{2}{3}\right)^{10}\right);$$

$$\text{Now } \left(\frac{2}{3}\right)^{10}=\frac{2^{10}}{3^{10}}=\frac{1024}{59049};$$

$$\therefore\ 1-\left(\frac{2}{3}\right)^{10}=1-\frac{1024}{59049}=\frac{58025}{59049},$$

$$\text{and } S=\frac{3\times 58025}{59049}=\frac{174075}{59049}.$$

Ex. 3. Find the sum of 1, 2, 4, 8, 16, &c. to 14 terms.

ANSWER, 16383.

Ex. 4. . . . . $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, &c. to 8 terms.

ANSW. $\frac{3280}{2187}$.

XXXIV.

On the method of finding any number of Arithmetic or Geometric Means between two numbers.

112. Let l be the last term of an arithmetic series, whose *first* term is (a), *common difference* (b), and *number of terms* (n); then $l=a+(n-1)b$; $\therefore\ (n-1)\,b=l-a$, or $b=\frac{l-a}{n-1}$. Now the number of *intermediate* terms between the first and the last is $n-2$; let $n-2=m$, then $n-1=m+1$. Hence $b=\frac{l-a}{m+1}$, which gives the following Rule for finding *any number* of arithmetic means between two numbers; "*Divide the difference of the two numbers by the given "number of means increased by unity, and the quotient will "be the common difference.*" Having the common difference, the *means* themselves will be known.

113. Let l be the last term of a *geometric* series, then $l=ar^{n-1}$, and $r^{n-1}=\frac{l}{a}$, $\therefore\ r=\sqrt[n-1]{\frac{l}{a}}$. The number of intermediate terms as before is $n-2$; let $n-2=m$, then $n-1=m+1$, and $r=\sqrt[m+1]{\frac{l}{a}}$, which gives the following rule

for finding *any number* of geometric means between two numbers; viz. "*Divide one number by the other, and take "that root of the quotient which is denoted by* $m+1$; *the "result will be the common ratio.*" Having the *common ratio*, the means are found by common multiplication.

EXAMPLE 1.

Find *six* arithmetic means between 1 and 43.

$$\text{Here }\left.\begin{array}{r} l=43 \\ a=\ 1 \\ m=\ 6 \end{array}\right\} \therefore b=\frac{l-a}{m+1}=\frac{43-1}{6+1}=\frac{42}{7}=6.$$

By adding this common difference continually to the lesser number (1), we have 7, 13, 19, 25, 31, 37, for the *six* means required.

Ex. 2.

Find three geometric means between 2 and 32.

$$\text{Here }\left.\begin{array}{r} a=\ 2 \\ l=32 \\ m=\ 3 \end{array}\right\} \therefore r=\sqrt[m+1]{\frac{l}{a}}=\sqrt[4]{\frac{32}{2}}=\sqrt[4]{16}=2,$$

and the means required are 4, 8, 16.

Ex. 3.

Find two geometric means between $\frac{16}{27}$ and 2.

$$\text{Here }\left.\begin{array}{r} a=\frac{16}{27} \\ l=2 \\ m=2 \end{array}\right\} \begin{array}{l} \therefore r=\sqrt[m+1]{\frac{l}{a}}=\sqrt[3]{2\times\frac{27}{16}}=\sqrt[3]{\frac{27}{8}}=\frac{3}{2}; \\ \qquad \therefore \text{ the two means are } \frac{8}{9} \text{ and } \frac{4}{3}. \end{array}$$

Ex. 4. Find seven arithmetic means between 3 and 59

ANSWER, 10, 17, 24, 31, 38, 45, 52.

Ex. 5. Find eight arithmetic means between 4 and 67.

Ex. 6. Find nine arithmetic means between 9 and 109.

Ex. 7. Find two geometric means between 4 and 256.

ANSW. 16 and 64.

Ex. 8. Find three geometric means between $\frac{1}{9}$ and 9.

ANSW. $\frac{1}{3}$, 1, 3.

114. Let $a, a+b, a+2b$, be three quantities in arithmetic progression, then the sum of the first and last $=2a+2b=2(a+b)$; $\therefore a+b=$ *half* the sum of the first and last; hence "an arithmetic mean between any *two* quantities is found, "by taking *half their sum*." Again, let a, ar, ar^2, be any three quantities in *geometric* progression, then the *product* of the first and last $=a^2r^2=$ the *square* of the mean term, from which it appears that "a *geometric* mean between "any two quantities is found by taking the *square root* "*of their product* (a)." From hence also it appears, that an *arithmetic* mean between any two numbers is greater than a *geometric* mean; for let the two numbers be $a+x$ and $a-x$, then the *arithmetic* mean is a and the *geometric* is $\sqrt{a^2-x^2}$, which is evidently less than a.

XXXV.

On the Solution of Equations relating to Numbers in Arithmetical or Geometrical Progression.

115. As the several terms of any arithmetic or geometric series may be expressed by means of *two unknown quantities*, it is not difficult to find the value of quantities of this kind, which shall bear such relations to each other as may be determined by *two equations*; of which the following are Examples.

EXAMPLE 1.

Find four numbers in arithmetical progression, such, that their *sum* shall be 56, and the *sum of their squares* 864.

Let $x=$ the *second* of these four numbers,
and $y=$ their *common difference*.

(a) It may be proper here to observe, that quantities which are in *geometric progression* are also in *continued proportion*; for $a : ar :: ar : ar^2 :: ar^2 : ar^3 ::$ &c. The *differences* of quantities in geometric progression are also in continued proportion; for the successive differences of the terms of the series a, ar, ar^2, ar^3, ar^1, &c. are $ar-a, ar^2-ar, ar^3-ar^2$, &c. or $ar-a, (ar-a)r, (ar-a)r^2$, &c. which is a geometric progression whose *first term* is $ar-a$, and *common ratio* r.

Then the four numbers may be represented by $x-y$, x, $x+y$, $x+2y$.

Hence, by the question,

$$(x-y)+x+(x+y)+(x+2y)=4x+2y=56,$$

and $$(x-y)^2+x^2+(x+y)^2+(x+2y)^2=4x^2+4xy+6y^2=864.$$

From 1st equation, $2x+y=28$.

Square this equation, then $4x^2+4xy+y^2=784$ (A),

but $4x^2+4xy+6y^2=864$ (B).

Subtract (A) from (B), and we have $5y^2=80$,

or $y^2=16$, and $y=4$;

$$\therefore x=\frac{28-y}{2}=\frac{24}{2}=12.$$

Hence 8, 12, 16, 20 are the four numbers required.

Ex. 2.

The *sum* of three numbers in arithmetic progression is 9, and the *sum of their cubes* is 153. What are the numbers?

Let $x-y$, x, $x+y$, be the numbers.

Then $(x-y)+x+(x+y)=3x=9$,

$(x-y)^3+x^3+(x+y)^3=3x^3+6xy^2=153$.

From 1st equation, $x=\frac{9}{3}=3$;

$\therefore$ by *substitution*, in 2d equation, $81+18y^2=153$,

or $18y^2=153-81=72$;

$\therefore y^2=\frac{72}{18}=4$, & $y=2$.

Hence, the numbers are 1, 3, 5.

Ex. 3.

Find three numbers in geometric progression, such, that their *sum* shall be equal to 7, and the *sum of their squares* to 21.

Let x, y, z, be the numbers.

Then, by the question, $x+y+z=7$, 1st equation.

And $x^2+y^2+z^2=21$, 2d equation.

By Note (*), Art. 114, $x : y :: y : z$; $\therefore y^2 = xz$.

From 1st equation, $x + z = 7 - y$.

Square this equation, and $x^2 + 2xz + z^2 = 49 - 14y + y^2 \; (A)$;

but $2xz = 2y^2 \; (B)$.

Subtract (B) from (A), then $x^2 + z^2 = 49 - 14y - y^2$.

But, from *second* equation, $x^2 + z^2 = 21 - y^2$.

Hence, $49 - 14y - y^2 = 21 - y^2$,

or $49 - 14y = 21$;

$\therefore 14y = 49 - 21 = 28$.

$$y = \frac{28}{14} = 2.$$

Again, since $x + z = 7 - y = 7 - 2 = 5$,

we have $x^2 + 2xz + z^2 = 25$;

but $4xz = 16$, for $xz = y^2$;

$\therefore$ by subtraction, $x^2 - 2xz + z^2 = 25 - 16 = 9$,

and $x - z = 3$.

$$\text{Hence, } \left.\begin{array}{r} x + z = 5, \\ x - z = 3; \end{array}\right\} \therefore \begin{array}{l} 2x = 8, \text{ or } x = 4, \\ 2z = 2, \text{ or } z = 1, \end{array}$$

and the three numbers are 1, 2, 4.

Ex. 4.

The *sum* of four numbers in geometric progression is 30, and the *last term divided by the sum of the mean terms* is $\frac{4}{3}$; What are the numbers?

$$\text{Let } \left.\begin{array}{l} x = \text{first term}, \\ y = \text{the } \textit{common ratio}; \end{array}\right\} \begin{array}{l} \text{then the numbers themselves} \\ \text{will be } x,\ xy,\ xy^2,\ xy^3. \end{array}$$

$$\text{Hence, by the question, } \left.\begin{array}{l} x + xy + xy^2 + xy^3 = 30, \text{ 1st equation,} \\ \text{and } \dfrac{xy^3}{xy + xy^2} = \dfrac{4}{3}, \text{ 2d equation.} \end{array}\right\}$$

$$\left.\begin{array}{l} \text{From 1st} \\ \text{equation,} \end{array}\right\} x \times (1 + y + y^2 + y^3) = 30, \text{ or } x = \frac{30}{1 + y + y^2 + y^3} \; (A).$$

$$\left.\begin{array}{l} \text{From 2d} \\ \text{equation,} \end{array}\right\} \frac{xy \times y^2}{xy \times (1 + y)} = \frac{4}{3}, \text{ or } \frac{y^2}{1 + y} = \frac{4}{3} \; (B).$$

By reduction of equation (*B*), $\Big\}\ 3y^2=4+4y$,

$$\text{or } y^2-\frac{4}{3}y=\frac{4}{3};$$

$$\therefore\ y^2-\frac{4}{3}y+\frac{4}{9}=\frac{4}{3}+\frac{4}{9}=\frac{16}{9},$$

$$\text{and } y-\frac{2}{3}=\frac{4}{3};\ \text{or } y=\frac{6}{3}=2.$$

Hence, from equation (*A*), $x=\dfrac{30}{1+2+4+8}=\dfrac{30}{15}=2.$

The four numbers are therefore 2, 4, 8, 16.

Ex. 5.

There are three numbers in geometric progression, whose *product* is 64, and *sum of their cubes* 584; What are the numbers?

Let the numbers be x, xy, xy^2.

Then, by the question, $x\times xy\times xy^2$, or $x^3y^3=64$, 1st equation,

and $x^3+x^3y^3+x^3y^6=584$, 2d equation.

From 1st equation, $y^3=\dfrac{64}{x^3}$, and $y^6=\dfrac{4096}{x^6}$.

By substitution, in 2d equation, $\Big\}\ x^3+64+\dfrac{4096}{x^3}=584.$

$$\text{Hence, } x^6+64x^3+4096=584x^3,$$

$$\text{or } x^6-520x^3\qquad=-4096.$$

Solve this equation by the Rule in Art. 84. $\Big\}$ . . . and $x^3=8$; or $x=2$.

$$\text{Now } y^3=\frac{64}{x^3}=\frac{64}{8}=8;\ \therefore\ y=2.$$

And the three numbers are 2, 4, 8.

Ex. 6. The sum of three numbers in arithmetic progression is 15; and the sum of the squares of the two extremes is 58. What are the numbers? Answer, 3, 5, 7.

Ex. 7. There are four numbers in arithmetic progression; the sum of the two extremes is 8, and the product of the means is 15. What are the numbers?

Answ. 1, 3, 5, 7.

Ex. 8. There are four numbers in arithmetic progression; the sum of the squares of the two means is 2, and the sum of the squares of the two extremes is 18. What are the numbers? Answ. -3, -1, 1, 3.

Ex. 9. There are three numbers in geometric progression, whose *sum* is 21, and *sum of their squares* 189. What are the numbers? Answ. 3, 6, 12.

Ex. 10. There are three numbers in geometric progression; the sum of the *first* and *last* is 52, and the *square of the mean* is 100. What are the numbers? Answ. 2, 10, 50.

Ex. 11. There are three numbers in geometric progression, whose sum is 31, and the sum of the *first* and *last* is 26. What are the numbers? Answ. 1, 5, 25.(a)

XXXVI.

On the Summation of an Infinite Series of Fractions in Geometric Progression; and on the method of finding the value of Circulating Decimals.

116. The general expression for the sum of a geometric series whose common ratio (r) is a *fraction*, is (Art. 110) $S=\frac{a-ar^n}{1-r}$. Suppose now n to *increase indefinitely*, then r^n (r being a proper fraction) will *decrease indefinitely*(b); therefore ar^n will decrease indefinitely with respect to a, or a will be the limit of $a-ar^n$, and $\frac{a}{1-r}$ the limit of $\frac{a-ar^n}{1-r}$ or S; and consequently $\frac{a}{1-r}$ will express the value of the series when the number of its terms is supposed to be indefinitely increased, or (as it is commonly called) the *sum of the series ad infinitum*.

(a) Some curious Theorems relating to numbers in *Geometrical* Progression will be found in "*Elémens d'Algèbre*, par l'Huilier," Vol. II. p. 177...208. Ed. 1812. A great variety of questions, both in *Arithmetical* and *Geometrical* Progression, will also be found in Bland's "*Algebraical Problems*."

(b) Let $r=\frac{1}{10}$, for instance; then $r^2=\frac{1}{100}$, $r^3=\frac{1}{1000}$, $r^4=\frac{1}{10000}$, &c.; from which it is evident, that if there be no limit to the increase of the index n, there will be none to the decrease of the fraction r^n.

Example 1.

Find the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, &c. *ad infinitum.*

$$\left.\begin{aligned}\text{Here } a&=1\\ r&=\frac{1}{2}\end{aligned}\right\}\therefore S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=\frac{2}{2-1}=2.$$

Ex. 2.

Find the value of $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+$ &c. *ad infinitum.*

$$\left.\begin{aligned}\text{Here } a&=\frac{1}{5}\\ r&=\frac{1}{5}\end{aligned}\right\}S=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{1}{5-1}=\frac{1}{4}.$$

Ex. 3.

Find the value of $\frac{3}{4}+\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+$ &c. *ad infinitum.*

$$\left.\begin{aligned}\text{Here } a&=\frac{3}{4}\\ r&=\frac{2}{3}\end{aligned}\right\}S=\frac{\frac{3}{4}}{1-\frac{2}{3}}=\frac{3}{4-\frac{8}{3}}=\frac{9}{12-8}=\frac{9}{4}.$$

Ex. 4. Find the value of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ &c. *ad infinitum.*

Answ. $\frac{3}{2}$.

Ex. 5. . . . . $\frac{5}{3}+1+\frac{3}{5}+\frac{9}{25}+$ &c. *ad infinitum.*

Answ. $4\frac{1}{6}$.

117. These operations furnish us with an expeditious method of finding the value of *circulating decimals,* the numbers composing which are geometric progressions,* whose *common ratios* are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. according to the number of factors contained in the *repeating* decimal.

EXAMPLE 1.

Find the value of the circulating decimal .33333, &c This decimal is represented by the geometric series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ &c. whose *first term* is $\frac{3}{10}$, and *common ratio* $\frac{1}{10}$.

$$\text{Hence }\left.\begin{aligned}a&=\frac{3}{10}\\ r&=\frac{1}{10}\end{aligned}\right\}\therefore S=\frac{a}{1-r}=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{3}{10-1}=\frac{3}{9}=\frac{1}{3}.$$

Ex. 2.

Find the value of .32323232, &c. ad *infinitum*.

$$\text{Here }\left.\begin{aligned}a&=\frac{32}{100}\\ r&=\frac{1}{100}\end{aligned}\right\}\therefore S=\frac{a}{1-r}=\frac{\frac{32}{100}}{1-\frac{1}{100}}=\frac{32}{100-1}=\frac{32}{99}.$$

Ex. 3.

Find the value of .713333, &c. *ad infinitum*.

The series of fractions representing the value of this decimal are

$$\frac{71}{100}+(\text{geometric series})\frac{3}{1000}+\frac{3}{10000}+\&\text{c.}=\frac{71}{100}+S.$$

$$\text{Here }\left.\begin{aligned}a&=\frac{3}{1000}\\ r&=\frac{1}{10}\end{aligned}\right\}\therefore S=\frac{\frac{3}{1000}}{1-\frac{1}{10}}=\frac{3}{1000-100}=\frac{3}{900}=\frac{1}{300}.$$

Hence the value of the decimal $=\left(\frac{71}{100}+S\right)\frac{71}{100}+\frac{1}{300}=\frac{214}{300}$ $=\frac{107}{150}$.

Ex. 4.

Find the value of .81343434, &c. *ad infinitum*.

$$\text{Here }\left.\begin{aligned}a&=\frac{34}{10000}\\ r&=\frac{1}{100}\end{aligned}\right\}\therefore S=\frac{a}{1-r}=\frac{\frac{34}{10000}}{1-\frac{1}{100}}=\frac{34}{10000-100}=\frac{34}{990}$$

And the value of the decimal $=\frac{81}{100}+S=\frac{81}{100}+\frac{34}{9900}=\frac{8053}{9900}$.

Ex. 5. Find the value of .77777, &c. *ad infinitum.*

ANSWER, $\frac{7}{9}$.

Ex. 6. Find the values of .232323 &c.; .83333 &c.; .7111411 &c.; and .956666 &c. *ad infinitum.*

ANSW. $\frac{23}{99}$; $\frac{5}{6}$; $\frac{707}{990}$; and $\frac{287}{300}$ respectively.

CHAP. VIII.

ON SURDS.

SURD Quantities have already been defined in Art. 55. and may be expressed either by the *radical sign*, or by their *fractional indices* (as in Art. 66.); thus the *square root of* 2, the *cube root of* 3, the *n*th *root of* $a+b$, the *cube root of* $(a+x)^2$, &c. &c. may be expressed either by $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[n]{a+b}$, $\sqrt[3]{(a+x)^2}$, &c. or by $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $(a+b)^{\frac{1}{n}}$, $(a+x)^{\frac{2}{3}}$, &c.

The precise value of these quantities cannot be ascertained; it can only be expressed by means of *decimals* or *series* which do not terminate; and in this sense they are called *irrational*, to distinguish them from all other quantities whatever, integral or fractional, whose values are determinate, and which are therefore denominated *rational.*

XXXVII.

ON THE REDUCTION OF SURDS.

CASE 1.

118. *A* RATIONAL *quantity may be reduced to the form of a surd, by raising it to the power denoted by the root of the surd, and then annexing the radical sign.*

EXAMPLE 1.

Reduce 3 to the form of the *square root*, and it becomes $\sqrt{3^2}$ or $\sqrt{9}$.

Ex. 2.

Reduce $\frac{2}{3}$ . . . . . *cube root* : . . . . $\sqrt[3]{\frac{2^3}{3^3}}$ or $\sqrt[3]{\frac{8}{27}}$.

Ex. 3.

Reduce $a+b$ . . . *square root* . . . . $\sqrt{(a+b)^2}$.

Ex. 4.

Reduce $4b^{\frac{2}{3}}$ . . . . *cube root* . . . . . $\sqrt[3]{64b^2}$.

CASE II.

119. *Surds of different indices are reduced to equivalent ones having the same radical sign, by bringing their fractional indices to a common denominator.*

Ex. 1. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to surds of the same radical sign. The fractions $\frac{1}{2}$, and $\frac{1}{3}$, reduced to a common denominator, are $\frac{3}{6}$ and $\frac{2}{6}$;

$$\left.\begin{array}{l} \therefore\ a^{\frac{1}{2}}=a^{\frac{3}{6}}=\sqrt[6]{a^3}, \\ \text{and } a^{\frac{1}{3}}=a^{\frac{2}{6}}=\sqrt[6]{a^2}; \end{array}\right\} \text{which are surds with the same radical sign.}$$

Ex. 2. Reduce $3^{\frac{2}{3}}$ and $5^{\frac{1}{2}}$ to surds of the same radical sign. The fractions $\frac{2}{3}$ and $\frac{1}{2}$, reduced to a common denominator, are $\frac{4}{6}$ and $\frac{3}{6}$.

Now $3^{\frac{4}{6}}=\sqrt[6]{3^4}=\sqrt[6]{81}$; and $5^{\frac{3}{6}}=\sqrt[6]{5^3}=\sqrt[6]{125}$.

Ex. 3. Reduce	$a^{\frac{1}{3}}$ and $b^{\frac{1}{5}}$	to Surds with the *same* radical sign.	ANS. $\sqrt{a^5}$ and $\sqrt[15]{b^3}$.
Ex. 4.	$c^{\frac{3}{4}}$ and $d^{\frac{1}{2}}$		. . . $\sqrt[4]{c^3}$ and $\sqrt[4]{d^2}$.
Ex. 5.	$3\sqrt[3]{2}$ & $2\sqrt{5}$		. . . $3\sqrt[6]{4}$ & $2\sqrt[6]{125}$.
Ex. 6.	$4^{\frac{2}{3}}$ and $15^{\frac{1}{2}}$		. . . $\sqrt[6]{256}$ & $\sqrt[6]{3375}$.

CASE

CASE III.

120. *Surds are reduced to their simplest form, by observing whether the quantity under the radical sign contains, as a factor, a power corresponding to the given surd root and then extracting the root.*

EXAMPLES.

Ex. 1. $\sqrt{a^2b}=\sqrt{a^2}\times\sqrt{b}=a\sqrt{b}$.

Ex. 2. $\sqrt[m]{a^mx}=\sqrt[m]{a^m}\times\sqrt[m]{x}=a\sqrt[m]{x}$.

Ex. 3. $\sqrt{72}=\sqrt{36\times2}=\sqrt{36}\times\sqrt{2}=6\sqrt{2}$.

Ex. 4. $\sqrt[3]{108}=\sqrt[3]{27\times4}=\sqrt[3]{27}\times\sqrt[3]{4}=3\sqrt[3]{4}$.

Ex. 5. $\sqrt[3]{2a^3b^2+a^5bc}=\sqrt[3]{a^3(2b^2+a^2bc)}$.

$$=\sqrt[3]{a^3}\times\sqrt[3]{2b^2+a^2bc}.$$

$$=\quad a\sqrt[3]{2b^2+a^2bc}.$$

Ex. 6. Reduce $\sqrt{a^4bc}$ & $\sqrt{98a^2x}$ to their simplest form. ANS. $a^2\sqrt{bc}$ & $7a\sqrt{2x}$.

Ex. 7. . . . . . $\sqrt[3]{a^3+a^3b^2}$ . . . $a\sqrt[3]{1+b^2}$.

Ex. 8. . . . . . $\sqrt{56}$ and $\sqrt[3]{72}$ . . . $2\sqrt{14}$ and $2\sqrt[3]{9}$.

Ex. 9. . . . . . $\sqrt[4]{243}$ and $\sqrt[5]{96}$ . . . $3\sqrt[4]{3}$ and $2\sqrt[5]{3}$.

The quantity *without* the radical sign is called the *coefficient* of the surd; and it is evident, that this quantity may always be put *under* the radical sign, by raising it to the power denoted by the index of the surd.

Thus, $7a\sqrt{2x}=$ (by Case I.) $\sqrt{7a\times7a}\times\sqrt{2x}$.

$$=\sqrt{49a^2}\times\sqrt{2x}=\sqrt{98a^2x}.$$

Also; $x\sqrt{2a-x}=\sqrt{x^2}\times\sqrt{2a-x}$.

$$=\sqrt{x^2(2a-x)}=\sqrt{2ax^2-x^3}.$$

CASE IV.

121. If the quantity under the radical sign be a *fraction*, it may be reduced to an *integral* form by the following process.

Multiply the numerator and denominator of the fraction by such a quantity as will make the denominator a complete power, corresponding to the root; and then proceed as in CASE III.

Ex. 1. $$\frac{c}{d}\times\sqrt{\frac{a^2}{b}}=\frac{c}{d}\times\sqrt{\frac{ab}{b^2}}.$$

$$=\frac{c}{d}\times\sqrt{\frac{a^2}{b^2}}\times\sqrt{b}.$$

$$=\frac{c}{d}\times\frac{a}{b}\times\sqrt{b}=\frac{ac}{bd}\sqrt{b}.$$

Ex. 2. $$\frac{3}{4}\times\sqrt{\frac{2}{7}}=\frac{3}{4}\times\sqrt{\frac{2\times7}{7\times7}}.$$

$$=\frac{3}{4}\sqrt{\frac{1}{49}\times14}.$$

$$=\frac{3}{4}\sqrt{\frac{1}{49}}\times\sqrt{14}.$$

$$=\frac{3}{4}\times\frac{1}{7}\times\sqrt{14}=\frac{3}{28}\sqrt{14}.$$

Ex. 3. $$\frac{1}{3}\sqrt[3]{\frac{16}{81}}=\frac{1}{3}\sqrt[3]{\frac{8\times2}{27\times3}}=\frac{1}{3}\times\frac{2}{3}\times\sqrt[3]{\frac{2}{3}}.$$

$$=\frac{2}{9}\times\sqrt[3]{\frac{2}{3}}.$$

$$=\frac{2}{9}\times\sqrt[3]{\frac{2\times3^2}{3^3}}.$$

$$=\frac{2}{9}\times\sqrt[3]{\frac{1}{27}\times18}.$$

$$=\frac{2}{9}\times\frac{1}{3}\times\sqrt[3]{18}=\frac{2}{27}\sqrt[3]{18}.$$

Ex. 4.

Reduce $x\sqrt{\frac{b}{y}}$ and $a\sqrt[3]{\frac{c^2}{a}}$ } to *integral* Surds in their simplest form. { Ans. $\frac{x}{y}\sqrt{by}$ and $\sqrt[3]{c^2a^2}$.

Ex. 5.

. . . . . $\sqrt{\frac{50}{147}}$ and $2\sqrt[3]{\frac{3}{4}}$ . . . $\frac{5}{21}\sqrt{6}$ and $\sqrt[3]{6}$.

XXXVIII.

On the application of the Fundamental Rules of Arithmetic to Surd Quantities.

122. *On the Addition and Subtraction of Surds.*

RULE.—*Reduce them to their simplest form; and if the surd part be the same in both, then their sum or difference will be found by taking the sum or difference of their coefficients.*

EXAMPLE 1.

Find the *sum* and *difference* of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$.

By ART. 120, $\sqrt{16a^2x}=4a\sqrt{x}$,

and $\sqrt{4a^2x}=2a\sqrt{x}$;

$\therefore$ the *sum* $=4a\sqrt{x}+2a\sqrt{x}=(4a+2a)\times\sqrt{x}=6a\sqrt{x}$.

the *difference* $=4a\sqrt{x}-2a\sqrt{x}=(4a-2a)\times\sqrt{x}=2a\sqrt{x}$.

Ex. 2.

Find the *sum* and *difference* of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

By ART. 120, $\sqrt[3]{192}=\sqrt[3]{64\times3}=4\sqrt[3]{3}$,

and $\sqrt[3]{24}=\sqrt[3]{8\times3}=2\sqrt[3]{3}$;

$\therefore \sqrt[3]{192}\pm\sqrt[3]{24}=(4\pm2)\sqrt[3]{3}=6\sqrt[3]{3}$ or $2\sqrt[3]{3}$.

Ex. 3.

Find the *sum* and *difference* of $\sqrt{\frac{8}{27}}$ and $\sqrt{\frac{1}{6}}$.

The two fractions $\frac{8}{27}$ and $\frac{1}{6}$, reduced to a *common denominator*, are $\frac{48}{162}$ and $\frac{27}{162}$.

$$\text{Now } \sqrt{\frac{48}{162}}=\sqrt{\frac{16\times3}{81\times2}}=\frac{4}{9}\sqrt{\frac{3}{2}},$$

$$\text{and } \sqrt{\frac{27}{162}}=\sqrt{\frac{9\times3}{81\times2}}=\frac{3}{9}\sqrt{\frac{3}{2}}.$$

$$\text{Hence } \sqrt{\frac{8}{27}}\pm\sqrt{\frac{1}{6}}=\left(\frac{4}{9}\pm\frac{3}{9}\right)\sqrt{\frac{3}{2}}=\frac{7}{9}\sqrt{\frac{3}{2}}, \text{ or } \frac{1}{9}\sqrt{\frac{3}{2}}.$$

If the surd part be not the *same* in the quantities to be added or subtracted from each other, it is evident that such addition or subtraction can only be performed by placing the signs + or − between them.

Ex. 4. Add $\sqrt{27a^4x}$ and $\sqrt{3a^4x}$ together . ANSW. $4a^2\sqrt{3x}$.

Ex. 5. . . . $\sqrt{128}$ and $\sqrt{72}$ . . . . $14\sqrt{2}$.

Ex. 6. . . . $\sqrt[3]{135}$ and $\sqrt[3]{40}$ . . . . $5\sqrt[3]{5}$.

Ex. 7. Subtract $3\sqrt{\frac{5}{27}}$ from $4\sqrt{\frac{3}{5}}$ . . . . . . $\frac{7}{15}\sqrt{15}$.

Ex. 8. . . . . . . $\sqrt[3]{108}$ from $9\sqrt[3]{4}$ . . . . $6\sqrt[3]{4}$.

123. *On the Multiplication and Division of Surds.*

RULE.—*Reduce them, if necessary, to equivalent ones with the same index, and then multiply or divide both the rational and irrational parts respectively.*

EXAMPLE 1.

Multiply $\sqrt{a}$ by $\sqrt[3]{b}$, or $a^{\frac{1}{2}}$ by $b^{\frac{1}{3}}$.

The fractions $\frac{1}{2}$ and $\frac{1}{3}$, reduced to *common denominators*, are $\frac{3}{6}$, and $\frac{2}{6}$;

$$\therefore\ a^{\frac{1}{2}}=a^{\frac{3}{6}}=\sqrt[6]{a^3};\ \text{and}\ b^{\frac{1}{3}}=b^{\frac{2}{6}}=\sqrt[6]{b^2}.$$

$$\text{Hence}\ \sqrt{a}\times\sqrt[3]{b}=\sqrt[6]{a^3}\times\sqrt[6]{b^2}=\sqrt[6]{a^3b^2}.$$

Ex. 2.

Multiply $5\sqrt{5}$ by $3\sqrt{8}$.

$$5\sqrt{5}\times3\sqrt{8}=15\sqrt{40}=15\sqrt{4\times10}.$$
$$=15\times2\times\sqrt{10}=30\sqrt{10}.$$

Ex. 3.

Multiply $2\sqrt{3}$ by $3\sqrt[3]{4}$.

By *reduction*, $2\sqrt{3}=2\times3^{\frac{3}{6}}=2\times\sqrt[6]{3^3}=2\sqrt[6]{27}$,

and $3\sqrt[3]{4}=3\times4^{\frac{2}{6}}=3\times\sqrt[6]{4^2}=3\sqrt[6]{16}$.

Hence $2\sqrt{3}\times3\sqrt[3]{4}=2\sqrt[6]{27}\times3\sqrt[6]{16}=6\sqrt[6]{432}$

Ex. 4.

Divide $2\sqrt[3]{bc}$ by $3\sqrt{ac}$.

$$2\sqrt[3]{bc}=2\times(bc)^{\frac{2}{6}}=2\sqrt[6]{b^2c^2},$$

and $3\sqrt{ac}=3\times(ac)^{\frac{3}{6}}=3\sqrt[6]{a^3c^3}$;

$$\therefore \frac{2\sqrt[3]{bc}}{3\sqrt{ac}}=\frac{2}{3}\times\sqrt[6]{\frac{b^2c^2}{a^3c^3}}=\frac{2}{3}\sqrt[6]{\frac{b^2}{a^3c}}.$$

Ex. 5.

Divide $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$.

$$10\sqrt[3]{108}=10\sqrt[3]{27\times4}=10\times3\times\sqrt[3]{4}=30\sqrt[3]{4};$$

$$\therefore \frac{10\sqrt[3]{108}}{5\sqrt[3]{4}}=\frac{30\sqrt[3]{4}}{5\sqrt[3]{4}}=6;\text{ or thus; }\frac{10\sqrt[3]{108}}{5\sqrt[3]{4}}=2\sqrt[3]{27}.$$
$[=2\times3=6.$

Ex. 6. Multiply $\sqrt[3]{15}$ by $\sqrt{10}$ . . . ANSWER, $\sqrt[6]{225000}$.

Ex. 7. . . . . . . $\frac{1}{2}\sqrt[3]{6}$ by $\frac{2}{3}\sqrt[3]{18}$ . . . . $\sqrt[3]{4}$.

Ex. 8. Divide $10\sqrt{27}$ by $2\sqrt{3}$ . . . . 15.

Ex. 9. . . . . $10\sqrt[3]{108}$ by $5\sqrt[3]{84}$ . . . . . . $\frac{2}{7}\sqrt[3]{441}$.

124. *On the Involution and Evolution of Surds.*

RULE. *Raise the rational part to the power or root required, and then multiply the fractional index of the surd part by the index of that power or root.*

Ex. 1. The *square* of $\sqrt[3]{a}=a^{\frac{1}{3}\times2}=a^{\frac{2}{3}}=\sqrt[3]{a^2}$.

Ex. 2. *Cube* of $\sqrt[5]{b^2}=b^{\frac{2}{5}\times3}=b^{\frac{6}{5}}=\sqrt[5]{b^6}=b\sqrt[5]{b}$.

Ex. 3. *4th power* of $2\sqrt[3]{2}=16\times2^{\frac{1}{3}\times4}=16\times2^{\frac{4}{3}}=16\sqrt[3]{16}$
$[=32\sqrt[3]{2}$.

Ex. 4. *Square root* of $a^{\frac{1}{3}}b^{\frac{1}{2}}=a^{\frac{1}{3}\times\frac{1}{2}}b^{\frac{1}{2}\times\frac{1}{2}}=a^{\frac{1}{6}}b^{\frac{1}{4}}$.

Ex. 5. *Cube root* of $\frac{1}{8}\sqrt{2}=\frac{1}{2}\times2^{\frac{1}{2}\times\frac{1}{3}}=\frac{1}{2}\times2^{\frac{1}{6}}=\frac{1}{2}\sqrt[6]{2}$.

Ex. 6. Cube $\frac{1}{2}\sqrt{3}$ . . . . ANSWER, $\frac{3}{8}\sqrt{3}$.

Ex. 7. Find fourth power of $\frac{1}{6}\sqrt{6}$ . . . . . . $\frac{1}{36}$

Ex. 8. Find square root of $9\sqrt[3]{3}$ . . Answ. $3\sqrt[6]{3}$.

Ex. 9. . . . fourth root of $\frac{16}{81}\sqrt[3]{a^2}$ . . . . $\frac{2}{3}\sqrt[6]{a}$.

Ex. 10. . . . fifth root of $\frac{1}{32}\times\left(\frac{b^2}{a}\right)^3$ . . . . $\frac{\sqrt[5]{b^6}}{2\sqrt[5]{a^3}}$.

125. From the preceding rules we easily deduce the method of converting fractions whose denominators are *surd* quantities, into others whose denominators shall be *rational*. Thus, let both the numerator and denominator of the fraction $\frac{a}{\sqrt{x}}$ be multiplied by $\sqrt{x}$, and it becomes $\frac{a\sqrt{x}}{x}$; and by multiplying the numerator and denominator of the fraction $\frac{b}{\sqrt[3]{a+x}}$ by $\sqrt[3]{(a+x)^2}$ or $(a+x)^{\frac{2}{3}}$, it becomes $\frac{b(a+x)^{\frac{2}{3}}}{\sqrt[3]{(a+x)^3}}=\frac{b(a+x)^{\frac{2}{3}}}{a+x}$. Or in general, if both the numerator and denominator of a fraction of the form $\frac{a}{\sqrt[n]{x}}$ be multiplied by $\sqrt[n]{x^{n-1}}$, it becomes $\frac{a\sqrt[n]{x^{n-1}}}{x}$, a fraction whose denominator is a *rational* quantity.

XXXIX.

On the method of finding Multipliers which shall render Binomial Surd Quantities Rational.

126. *Compound* surd quantities are such as consist of two or more terms, some or all of which are *irrational;* and if a quantity of this kind consist only of *two* terms, it is called a *binomial* surd. The rule for finding a multiplier which shall render a binomial surd quantity *rational*, is derived from observing the quotient which arises from the actual division of the numerator of the following fractions by the denominator. Thus,

I. $\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2+\&c.\ldots+y^{n-1}$ to n terms, whether n be *even* or *odd*.

II. $\frac{x^n-y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\&c.\ldots-y^{n-1}$ to n terms, when n is an *even* number.

III. $\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\&c.\ldots+y^{n-1}$ to n terms, when n is an *odd* number[a].

127. Now let $x^n=a$, $y^n=b$, then $x=\sqrt[n]{a}$, $y=\sqrt[n]{b}$, and these fractions severally become $\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}}$, $\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}}$, and $\frac{a+b}{\sqrt[n]{a}+\sqrt[n]{b}}$; and by the application of the foregoing rules we have $x^{n-1}=\sqrt[n]{a^{n-1}}$; $x^{n-2}=\sqrt[n]{a^{n-2}}$; $x^{n-3}=\sqrt[n]{a^{n-3}}$, &c.; also, $y^2=\sqrt[n]{b^2}$; $y^3=\sqrt[n]{b^3}$; &c.; hence, $x^{n-2}y=\sqrt[n]{a^{n-2}}\times\sqrt[n]{b}=\sqrt[n]{a^{n-2}b}$; $x^{n-3}y^2=\sqrt[n]{a^{n-3}}\times\sqrt[n]{b^2}=\sqrt[n]{a^{n-3}b^2}$; &c. By substituting these values of x^{n-1}, $x^{n-2}y$, $x^{n-3}y^2$, &c. in the several quotients, we have

$$\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\&c.\ \ldots+\sqrt[n]{b^{n-1}}\text{ to}$$

n terms; where n may be any whole number whatever.

And

$$\frac{a\pm b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\&c.\ \ldots\pm\sqrt[n]{b^{n-1}}\text{ to}$$

n terms; where the terms b and $\sqrt[n]{b^{n-1}}$ have the sign $+$, when n is an *odd* number; and the sign $-$, when n is an *even* number.

128. Since the *divisor* multiplied by the *quotient* gives the *dividend*, it appears from the foregoing operations that

---

[a] For I. $\frac{x^2-y^2}{x-y}=x+y$; $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$; $\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3$; &c.

II. $\frac{x^2-y^2}{x+y}=x-y$; $\frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3$; &c.

III. $\frac{x+y}{x+y}=1$; $\frac{x^3+y^3}{x+y}=x^2-xy+y^2$; $\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4$; &c.

"if a binomial surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$ be multiplied by "$\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+$ &c. . . . $+\sqrt[n]{b^{n-1}}$ (n being any "whole number whatever), the product will be $a-b$, a "*rational* quantity; and if a binomial surd of the form "$\sqrt[n]{a}+\sqrt[n]{b}$ be multiplied by $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-$ &c. ". . . . . $\pm\sqrt[n]{b^{n-1}}$, the product will be $a+b$ or $a-b$, according as the index n is an *odd* or an *even* number." The great use of this rule is, "to convert fractions having "*surd* denominators, into others which shall have *rational* "ones;" of which the following are examples.

EXAMPLE 1.

Reduce $\frac{x}{a-\sqrt{x}}$ and $\frac{\sqrt{6}}{\sqrt{8}+\sqrt{3}}$ to fractions having rational denominators.

Since "the *sum* into the *difference* of two quantities gives the *difference of their squares*," it is evident that these fractions may be reduced to others having *rational* denominators, by multiplying their numerators and denominators by $a+\sqrt{x}$ and $\sqrt{8}-\sqrt{3}$ respectively, without the formal application of the rule.

$$\left.\begin{array}{l}\text{Thus } x(a+\sqrt{x})=ax+x\sqrt{x}\\ \text{and}(a-\sqrt{x})(a+\sqrt{x})=a^2-x\end{array}\right\}$$ by which means the fraction is reduced to $\frac{ax+x\sqrt{x}}{a^2-x}$.

Again $\sqrt{6}(\sqrt{8}-\sqrt{3})=\sqrt{48}-\sqrt{18}=$ (Art. 120.) $4\sqrt{3}-3\sqrt{2}$, and $(\sqrt{8}+\sqrt{3})(\sqrt{8}-\sqrt{3})=8-3=5$;

and the fraction is reduced to $\frac{4\sqrt{3}-3\sqrt{2}}{5}$

Ex. 2.

Reduce $\frac{2}{\sqrt[3]{3}-\sqrt[3]{2}}$ to a fraction with a rational denominator.

To find the multiplier which shall make $\sqrt[3]{3}-\sqrt[3]{2}$ *rational*, we have $n=3$, $a=3$, $b=2$; $\therefore \sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{b^{n-1}}$[a] $=\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$. Now

---

(a) The *number* of terms of the general series to be taken, is always equal to n; in the present instance, therefore, the number to be taken is 3; and so in all other cases; recollecting that the *last* term is always $\sqrt[n]{b^{n-1}}$.

Now $2(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4})=2\sqrt[3]{9}+2\sqrt[3]{6}=2\sqrt[3]{4}$, and $(\sqrt[3]{3}-\sqrt[3]{2})(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4})=(a-b)3-2=1$; ∴ the *denominator* is 1, and the fraction is reduced to $2\sqrt[3]{9}+2\sqrt[3]{6}+2\sqrt[3]{4}$.

Ex. 3.

Reduce $\frac{c}{\sqrt[3]{x}+\sqrt[3]{y}}$ to a fraction with a rational denominator.

Here $n=3$, $a=x$, $b=y$, the sign of $\sqrt[n]{b}$ is +, and n an *odd* number; ∴ the multiplier is $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{b^{n-1}}=\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}$; Hence the fraction required is $\left(\frac{c}{\sqrt[3]{x}+\sqrt[3]{y}}\right)$

$$\left(\frac{\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}}{\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}}\right)=\frac{c}{x+y}\left(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}\right).$$

Ex. 4.

Reduce $\frac{3}{\sqrt[4]{5}+\sqrt[4]{3}}$ to a fraction with a rational dénominator.

Here $n=4$, $a=5$, $b=3$, the sign of $\sqrt[n]{b}$ is +, and n an even number, ∴ the multiplier is $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{b^{n-1}}=\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}$. Hence the fraction required is $\left(\frac{3}{\sqrt[4]{5}+\sqrt[4]{3}}\right)$

$$\left(\frac{\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}}{\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}}\right)=\frac{3}{2}(\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}.)$$

XL.

On the Method of extracting the Square Root of Binomial Surds.

129. Let $\sqrt{x}$ and $\sqrt{y}$ be two quadratic surds, which are not reducible to the *same irrational part*; their product will be irrational. For, if $\sqrt{x}\times\sqrt{y}=m$, $\sqrt{x}=\frac{m}{\sqrt{y}}=\frac{m}{y}\sqrt{y}$; that is, $\sqrt{x}$ is reducible to the irrational part $\sqrt{y}$, contrary to the supposition.

130. Next, let $\sqrt{x}+\sqrt{y}$ be a binomial, both whose terms are quadratic surds, not reducible to the same irrational part. If this binomial be squared, the result is $x+y+2\sqrt{xy}$, a quantity of which one part is rational, and the other (Art. 129.) irrational. Let $x+y=a$ and $2\sqrt{xy}=\sqrt{b}$, then it appears that every binomial surd whose square root can be exhibited under the form of $\sqrt{x}+\sqrt{y}$ must be of the form $a+\sqrt{b}$; a being a rational quantity and $\sqrt{b}$ a quadratic surd. The same will evidently be true, if *one* of the terms, as $\sqrt{x}$, be supposed *rational.*

131. The square root of a rational quantity cannot be partly *rational* and partly a quadratic surd. For, if possible, let $\sqrt{x}=a\pm\sqrt{b}$; then $x=a^2+b\pm 2a\sqrt{b}$, and $\sqrt{b}=\dfrac{x-a^2-b}{\pm 2a}$, a *rational* quantity. But by the supposition, $\sqrt{b}$ is a *surd;* hence $\sqrt{x}$ cannot be expressed under the form $a\pm\sqrt{b}$. In the same manner it may be proved, that the square root of a rational quantity cannot be equal to the *sum* or *difference* of two quadratic surds not reducible to the same irrational part. For, if possible, let $\sqrt{x}=\sqrt{a}\pm\sqrt{b}$, then $x=a+b\pm 2\sqrt{ab}$, and $\sqrt{ab}=\dfrac{x-a-b}{\pm 2}$, which is impossible by Art. 129.

132. In any equation $x+\sqrt{y}=a+\sqrt{b}$, consisting of rational quantities and quadratic surds, the *rational* parts on each side are equal, and also the *irrational.* For if x be not equal to a, let $x=a\pm m$, then $a\pm m+\sqrt{y}=a+\sqrt{b}$, or $\pm m+\sqrt{y}=\sqrt{b}$, i.e. $\sqrt{b}$ is partly *rational* and partly *irrational,* which has already been proved to be impossible. In a similar manner it may be shewn, that in any equation $m\sqrt{x}+n\sqrt{y}=p\sqrt{x}+q\sqrt{y}$, where $\sqrt{x}$ and $\sqrt{y}$ cannot be reduced to the same irrational part, $m\sqrt{x}=p\sqrt{x}$, and $n\sqrt{y}=q\sqrt{y}$. For, if q be not equal to n, by transposition, $m\sqrt{x}=p\sqrt{x}+q\sqrt{y}-n\sqrt{y}=p\sqrt{x}+(q-n)\sqrt{y}$, contrary to Art. 131. $\therefore q\sqrt{y}=n\sqrt{y}$, and consequently $m\sqrt{x}=p\sqrt{x}$.

133. To find the square root of the binomial quadratic surd $a+\sqrt{b}$. Assume $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}$, then $x+y+2\sqrt{xy}=a+\sqrt{b}$; $\therefore$ (by Art. 132.) $x+y=a$, and $2\sqrt{xy}=\sqrt{b}$; hence $x+2xy+y^2=a^2$ (A), and $4xy=b$ (B); subtract (B) from (A), then $x^2-2xy+y^2=a^2-b$, and $x-y=\sqrt{a^2-b}$; we have, therefore,

$$\left.\begin{array}{l} x+y=a \\ x-y=\sqrt{a^2-b} \end{array}\right\} \therefore \begin{array}{l} 2x=a+\sqrt{a^2-b}, \text{ and } x=\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}. \\ 2y=a-\sqrt{a^2-b}, \text{ and } y=\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}. \end{array}$$

Hence $\sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}$, an expression which can evidently be of the form $\sqrt{x}+\sqrt{y}$, only when $\sqrt{a^2-b}$ is a *rational* quantity. The square root of the binomial surd quantity $a+\sqrt{b}$ can therefore be exhibited under the form $\sqrt{x}+\sqrt{y}$ only when a^2-b is a *square number*. By a similar process it might be shewn that the square root of $a\sqrt{b}$ is $\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}$, subject to the same limitation.

EXAMPLE 1.

What is the square root of $19+8\sqrt{3}$?

$$\left.\begin{array}{l} \text{Here } a=19 \\ \sqrt{b}=8\sqrt{3} \end{array}\right\} \therefore a^2-b=361-192=169, \text{ and } \sqrt{a^2-b}=13.$$

$$\text{Hence } \sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\frac{19}{2}+\frac{13}{2}}+\sqrt{\frac{19}{2}-\frac{13}{2}}=\sqrt{16}+\sqrt{3}=4+\sqrt{3}.$$

Ex. 2.

Find the square root of $12-\sqrt{140}$.

$$\left.\begin{array}{l} \text{Here } a=12 \\ \sqrt{b}=\sqrt{140} \end{array}\right\} \therefore a^2-b=144-140=4, \text{ and } \sqrt{a^2-b}=2.$$

$$\text{Hence } \sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{6+1}-\sqrt{6-1}=\sqrt{7}-\sqrt{5}.$$

Ex. 3.

Find the square root of $31+12\sqrt{-5}$.

Here

$$\left.\begin{array}{l}\text{Here } a=31\\ \quad\sqrt{b}=12\sqrt{-5}\\ \quad\text{or } b=-720\end{array}\right\}\therefore a^2-b=961+720=1681,\text{ and }\sqrt{a^2-b}=41.$$

$$\text{Hence }\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\frac{31}{2}+\frac{41}{2}}+\sqrt{\frac{31}{2}-\frac{41}{2}}=6+\sqrt{-5}.$$

Ex. 4. Find the square root of $7+4\sqrt{3}$ ... Answ. $2+\sqrt{3}$.

Ex. 5. . . . . $7-2\sqrt{10}$ . . . $\sqrt{5}-\sqrt{2}$.

Ex. 6. . . . . $18-10\sqrt{-7}$ . . . $5-\sqrt{-7}$

CHAP. IX.

ON MISCELLANEOUS SUBJECTS.

We now proceed to apply the principles laid down in the preceding Chapters to the investigation of questions of a miscellaneous nature, beginning with some observations upon *prime* numbers and their several relations.

XLI.

On Prime Numbers and their Relations; and on the Method of finding the least Common Multiple of two or more Numbers.

134. Numbers which admit of no exact divisor, or which have no measure (Art. 40), except themselves and unity, are called *Prime Numbers,* as 2, 3, 5, 7, &c.; and two or more numbers, which have no common divisor, or measure, greater than unity, are said to be *prime to each other,* as 8 and 9; 11, 14, and 15; &c.

135. "Let ab, the product of any two numbers, be divisible by c; then, if c be prime to b, it will be a divisor of a." For

suppose b to be greater than c; then, if the operation in Art. 45 be performed on them, the last divisor, or greatest common measure, will be unity, because b and c are prime to each other. Let the operation stand as follows;

$$\left.\begin{array}{l} c)b(p \\ \;\;\underline{cp} \\ \;\;\;d)c(q \\ \;\;\;\;\;\underline{dq} \\ \;\;\;\;\;\;e)d(r \\ \;\;\;\;\;\;\;\;\underline{er} \\ \;\;\;\;\;\;\;\;\;\underline{1} \end{array}\right\} \begin{array}{l} \text{then we have these equations;} \\ \left.\begin{array}{l} b-cp=d \\ c-dq=e \\ d-er=1 \end{array}\right\} \text{ or } \left\{\begin{array}{l} ab-acp=ad \\ ac-adq=ae \\ ad-aer=a. \end{array}\right. \end{array}$$

Consequently, since c, by supposition, measures ab, it will measure $ab-acp$, or ad; and $ac-adq$, or ae; and $ad-aer$, or a. (Art. 43, 44.)

If c be supposed greater than b, we shall, by a similar process, arrive at the same conclusion; which will be equally true, whatever be the number of divisions in the operation.

136. Hence it follows, that if the numerator and denominator of a fraction be prime to each other, there can exist no other equal fraction having its numerator and denominator respectively less than those of the first.

In the fraction $\frac{a}{b}$, let a be prime to b; and let $\frac{m}{n}$ be an equal fraction; then, since $\frac{a}{b}=\frac{m}{n}$, $m=\frac{an}{b}$. Consequently b will be a divisor of an; and since, by supposition, it is prime to a, it must (Art. 135) be a divisor of n, and therefore less than n. In the same manner it may be proved that a is less than m, and the fraction $\frac{a}{b}$ is therefore in its least possible terms.

Again, since b is a divisor of n, let $\frac{n}{b}=p$; then $n=pb$, and consequently, since $\frac{pa}{pb}=\frac{a}{b}=\frac{m}{n}$, m will $=pa$; that is, "if "two fractions, of which the former is in its least terms, "be equal, the numerator and denominator of the latter "will be *equimultiples* of the numerator and denominator "of the former, respectively."

137. If a and b are both prime to c, ab will be prime to c. For if not, suppose ab and c to have a common measure m, and let $ab=mp$, and $c=mq$. Then, since a is prime to c, or mq, it is prime to m; for if a and m had a common measure, this would (Art. 43) be a common measure of a and mq. For the same reason, b is prime to m. But, since $ab=mp$, $\frac{a}{m}=\frac{p}{b}$, and $\frac{a}{m}$ (Art. 136) is in its lowest terms; therefore b is either equal to m, or (Art. 136) a multiple of m, which is absurd, because b has been proved to be prime to m; $\therefore$ ab and c can have no common measure, and consequently ab must be prime to c. In the same way, if a, b, c are all prime to d, abc is prime to d, and so on. Hence, if a be prime to d, a^2, a^3, a^4, &c. will all be prime to d.

Again, if a, b, c, &c. are each of them prime to each of d, e, f, &c. abc &c. will be prime to def &c. For, since a, b, c, &c. are prime to d, abc &c. will be prime to d. For the same reason, abc &c. is prime to e, f, &c., and consequently to def &c. Hence, if a be prime to d, a^2 will be prime to d^2, a^3 to d^3, and so on.

138. A *common multiple* of two or more numbers is any number which is measured by each of them; and their *least common multiple* is the least number which is so measured.

Let c be the greatest common measure of a and b, and let $a=mc$, $b=nc$. Then $ab=mnc^2$, and $\frac{ab}{c}=mnc=na=mb$; therefore $\frac{ab}{c}$ is a common multiple of a and b. It is also their *least* common multiple; for let d be any other common multiple of a and b, and let $d=pa=qb$; then $\frac{q}{p}=\frac{a}{b}=\frac{m}{n}$, where $\frac{m}{n}$ is in its least terms, because (c being the greatest common measure of a and b) m and n are prime to each other; therefore q and p are equimultiples (Art. 136) of m and n respectively, and q is greater than m; hence, qb is greater than mb, or d greater than $\frac{ab}{c}$. Hence, "the least "common multiple of two numbers is equal to their product "divided by their greater common measure." It may be farther observed, that "every other common multiple of a "and b is a multiple of their least common multiple;" for since q is a multiple of m, qb or d is a multiple of mb or $\frac{ab}{c}$.

To find the least common multiple of *three* numbers, "a, b, c; let m be the least common multiple of a and b, and "n the least common multiple of m and c; then n will be the least common multiple required." For since m is a common multiple of a and b, and n a common multiple of m and c, n will obviously be a common multiple of a, b, c. It will also be their *least* common multiple; for let d be any other multiple of a, b, c, then d will be a multiple of m, as has just been shewn; and since it is also a multiple of c, it will be a multiple of n, and therefore must be greater than n; hence n is the *least* common multiple of a, b, c.

XLII.

Properties of Numbers.

139. Let a, b, c, d, &c. represent the *digits* of a number, a being the digit in the *unit's* place, b the digit in the *ten's* place, c the digit in the *hundred's* place, &c. &c., and let $r = 10$, then the general value of any number may be represented by $a + br + cr^2 + dr^3 +$ &c.; thus, $357 = 7 + 50 + 300 = 7 + 5 \times 10 + 3 \times 10^2$; and $4213 = 3 + 1 \times 10 + 2 \times 10^2 + 4 \times 10^3$; &c. &c. From this mode of representing a number, the following properties are very readily deduced.

I. "*If from any number the sum of its digits be subtracted, the remainder* "*is divisible by* 9."

For let $a + br + cr^2 + dr^3 +$ &c. = the number.

Subtract $a + b + c + d +$ &c.

Then we have $b(r - 1) + c(r^2 - 1) + d(r^3 - 1) +$ &c. for the value of the number *when its digits are subtracted from it*; but, by Art. 126, this quantity is divisible by $r-1$ or 9. Take, for instance, the number 37591, subtract the *sum of its digits*, and the remainder is $37566=9\times4174$.

II. "*If the sum of the digits of any number be divisible by* 9, *the number itself is divisible by* 9." For let the number be N, and the *sum of its digits* S, and let $S=9m$. Then (by Property I.) $N-S$ is divisible by 9; let $N-S=9p$, and we have $N-9m=9p$, $\therefore N=9p+9m=9\,(p+m)$, which is divisible by 9; consequently N is divisible by 9. Thus the numbers 171, 387, 51489, &c., the sum of whose digits is divisible by 9, are *themselves* divisible by 9.

III. "*If the sum of the digits of any number be divisible by* 3, *then the number itself is divisible by* 3." Let N and S represent the *number* and *sum of its digits* as before, and let $S=3m$. Now $N-S=9p$, $\therefore N-3m=9p$, or $N=9p+3m$, which is evidently divisible by 3. Thus the numbers 111, 123, 258, 1713, &c. are all divisible by 3.

IV. "*If from any number the sum of the digits standing in the* ODD *places be* SUBTRACTED, *and to it the sum of the digits standing in the* EVEN *places be* ADDED, *then the result is divisible by* 11."

For let the number be $a+br+cr^2+dr^3+er^4+\&c.$

$$Add\;-a+b\;-c\;+d\;-e\;+\&c.$$

the *result* is $b.\overline{r+1}+c.\overline{r^2-1}+d.\overline{r^3+1}+e.\overline{r^4-1}+\&c.$; but by Art. 126, the quantities $r+1$, r^2-1, r^3+1, r^4-1, &c. are all divisible by $r+1$; therefore $b.\overline{r+1}+c.\overline{r^2-1}+d.\overline{r^3+1}+e.\overline{r^4-1}+\&c.$ is divisible by $r+1$, or $=11$. Take, for instance, the number 57937; *subtract* $5+9+7=21$, and *add* $7+3=10$, or, in other words, *subtract* 11, then the *remainder* $57926=11\times5266$.

V. "*If the sum of the digits standing in the* EVEN *places be equal to the sum of the digits standing in the* ODD *places, then the number is divisible by* 11." Let $N=$ the number, $S=$ the sum of the *even* digits, $s=$ the sum of the *odd* digits; then (IV.) $N+S-s$ is divisible by 11; but if $S=s$, then $S-s=0$, $\therefore N$ is divisible by 11. Thus the numbers 121363, 12133, 48422, &c. are all divisible by 11.

The number r (which is called the root of the scale) has here been supposed $=10$, that being its value in the common system of notation; but the above properties of numbers are true for any other system. For instance, if the system of notation be such that the value of the digits increase only in a *sixfold* instead of a *tenfold* proportion from the right to the left, then (since $r=6$, and consequently $r-1=5$, $r+1=7$)

what has just been proved with respect to the numbers 9 and 11, is equally true with respect to the numbers 5 and 7, in the system the root of whose scale is 6.

140. Suppose, now, that it was required to transform a number of the common arithmetical scale into another of the *same value*, where the root of the scale shall be r; let the given number be N, and let the digits of the number where the root of the scale is r, be a, b, c, d, &c.; then we have

$$N = a + br + cr^2 + dr^3 + \&c.$$

an equation in which N and r are given, to find the values of a, b, c, d, &c. Divide N by r, then the *quotient* is $b + cr + dr^2 + \&c.$ and the *remainder* a; divide $b + cr + dr^2 + \&c.$ by r, the *quotient* is $c + dr + \&c.$, and the *remainder* b; divide $c + dr + \&c.$ by r, the *quotient* is $d + \&c.$ and the *remainder* c; so that the rule is, "to divide the given number continually "by r till the last quotient is less than r, then this last quo- "tient, together with the several remainders taken in the "reverse order, will be digits of the number required." For instance, let it be required to convert the number 3714 into another number of the same value, wherein the value of each digit shall increase in a fourfold proportion from the right hand to the left. Here $r = 4$; and the operation will stand thus;

4)3714(2 = 1st remainder
4)928(0 = 2d D°.
4)232(0 = 3d D°.
4)58(2 = 4th D°.
4)14(2 = 5th D°.
3

Hence 322002, where the value of each digit increases in a *fourfold* proportion, is a Number of the same value with 3714, where the value of each digit increases in a *tenfold* proportion.

141. The foregoing properties of Numbers have been deduced from the manner in which they are represented by means of the series $a + br + cr^2 + dr^3 + \&c.$ But numbers may also be considered as arising from the *continued multiplication* of certain factors. The *most general* form under which numbers may be thus represented is $a^n\, b^m\, c^r\, d^s$ &c., where a, b, c, d, &c. are *prime* numbers, and n, m, r, s, &c. any

whole numbers whatever. One of the *simplest* cases of this kind is when the number comes under the form $a^n b$; and under this form we are enabled to investigate the expression for what is called a *perfect* number, i. e. "*a number which is equal to the sum of all its divisors.*"

The process is this. The divisors of $a^n b$ are $1, a, a^2; a^3$, &c. . . . a^n and b, ab, a^2b, a^3b, &c. . . . . . $a^{n-1}b$; hence by the supposition

$$a^n b = 1 + a + a^2 + a^3 + \&c + a^n + b + ab + a^2b + \&c. + a^{n-1}b$$

$$= (\text{by Art. 110.}) \ \frac{a^{n+1}-1}{a-1} + \frac{a^n b - b}{a-1}.$$

$$\therefore a^{n+1}b - a^n b = a^{n+1} - 1 + a^n b - b;$$

or $a^{n+1}b - 2a^n b + b = a^{n+1}-$[illegible], $\therefore b = \dfrac{a^{n+1}-1}{a^{n+1}-2a^n+1}$; but since b is a whole number, suppose $a^{n+1}-2a^n+1$ equal to *unity*, and consequently $a^{n+1}-2a^n=0$, or $a-2=0$, or $a=2$; hence $b=2^{n+1}-1$; and the expression $a^n b$ becomes $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ must be a *prime* number. Let $n=1, 2, 3, 4, 5, 6$, &c., then $2^{n+1}-1=3, 7, 15, 31, 63, 127, 255$, &c.; of which the *prime* numbers are 3, 7, 13, 127, &c., and the corresponding values of n are 1, 2, 4, 6, &c.; hence a system of *perfect* numbers may be generated in the following manner;

$$\left.\begin{array}{l} 2(2^2-1) = 2\times 3 = 6 \\ 2^2(2^3-1) = 4\times 7 = 28 \\ 2^4(2^5-1) = 16\times 31 = 496 \\ 2^6(2^2-1) = 64\times 127 = 8128 \end{array}\right\}$$

and proceeding in this manner, the *next* perfect number is found to be 33550336.

XLIII.

Permutations and Combinations.

142. By *Permutations* are meant the number of *changes* which any quantities *a, b, c, d,* &c. may undergo with respect to their order, when taken *two and two* together, *three and three*, &c. &c. Thus *ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc,* are the different permutations of the four quantities *a, b, c, d,* when taken *two and two* together; *abc, acb, bac, bca, cab, cba,* of the *three* quantities, *a, b, c,* when taken *three and three together*; &c. &c.

143. By *Combinations* are meant the number of *collections* which may be formed out of the quantities *a, b, c, d, e,* &c. taken *two and two* together, *three and three* together, &c. &c. without having regard to the *order* in which the quantities are arranged in each collection. Thus *ab, ac, ad, bc, bd, cd,*

are the *combinations* which can be formed out of the *four* quantities a, b, c, d, taken *two and two* together; $a\,b\,c$, $a\,b\,d$, $a\,c\,d$, $b\,c\,d$, the combinations which may be formed out of the same quantities, when taken *three and three* together, &c. &c.

144. Let there be n quantities, a, b, c, d, e, &c., taken *two and two* together; then, by Art. 142, it appears that there will be $(n-1)$ permutations in which a stands first; for the same reason there will be $(n-1)$ permutations in which b stands first; and so of c, d, e, &c. Hence there will be n times $(n-1)$ permutations of the form ab, ac, ad, ae, &c.; ba, bc, bd, be, &c.; ca, cb, cd, ce, &c.; i.e. "*the number of permutations of* n *things taken two and two is* $n(n-1)$."

145. If these n quantities be taken *three and three* together, then there will be $n(n-1)(n-2)$ *permutations*. For if $(n-1)$ be substituted for n in the last article, then the number of permutations of $n-1$ things taken *two and two* together will be $(n-1)(n-2)$; hence the number of permutations of b, c, d, e, &c., taken *two and two* together, are $(n-1)(n-2)$, and consequently there are $(n-1)(n-2)$ permutations of the quantities a, b, c, d, e, &c. taken *three and three* together, in which a may stand first; for the same reason there are $(n-1)(n-2)$ permutations in which b may stand first; and so of c, d, e, &c. The number of permutations of this kind will therefore amount to $n(n-1)(n-2)$.

146. In the same way it appears, that if the number of quantities be n, and they are taken m and m together, the number of permutations will be $n(n-1)(n-2)$ &c..... . $(n-m+1)$; and if $m=n$, i.e. if the permutations respect all the quantities at once, then (since $m-n=0$) the *number* of them will be $n(n-1)(n-2)$ &c.....2.1. Thus, the number of permutations which might be formed from the letters composing the word "*virtue*" are $6\times5\times4\times3\times2\times1=720$.

147. But if in this latter case the *same* letter should

occur any number of times, then it is evident that we must *divide* the whole number of permutations, by the number of times the permutations are *multiplied* by having *different* letters instead of the repetition of the same letter. Thus, if the same letter should occur *twice*, then we must divide by 2×1; if it should occur *thrice*, we must divide by $3 \times 2 \times 1$; if p *times*, by $1.2.3...p$; and so for any other letter which may occur more than once. Hence the general expression for the number of permutations of n things, of which there are p of *one* kind; r of *another*; q of *another*; &c. &c. is $\frac{n(n-1)(n-2)(n-3)...2.1}{1.2.3..p \times 1.2.3..r \times 1.2.3..q}$. Thus the permutations which may be formed by the letters composing the word "*easiness*" (since s occurs *thrice*, e *twice*) are $\frac{8.7.6.5.4.3.2.1}{1.2.3 \times 1.2} = 3360$.

148. From the expression (in Art. 146) for finding the number of *permutations* of n things taken m and m together, we immediately deduce the theorem for finding the number of *combinations* of n things taken in the same manner. For the permutations of n things taken *two and two* together being $n(n-1)$, and each *combination* admitting of as many *permutations* as may be made by *two* things (which is 2×1), the number of *combinations* must be equal to the number of *permutations* divided by 2; i. e. the number of *combinations* of n things taken *two and two* together is $\frac{n(n-1)}{2}$. For the same reason, the *combinations* of n things, taken *three and three* together, must be equal to $\frac{n(n-1)(n-2)}{1.2.3}$; and, in general, the combinations of n things taken m and m together must be equal to $\frac{n(n-1)(n-2)....(n-m+1)}{1.2.3....m}$.

XLIV.

Unlimited Problems.

149. It has already been observed (Art. 69), that in order

to obtain the solution of equations containing any number of unknown quantities, it is necessary that there should be as many equations as there are unknown quantities. If the number of equations be *less* than that of the unknown quantities, then the number of values of the unknown quantities will be *unlimited,* unless the problem be *limited* by circumstances. This will be readily understood by taking the simple case of $x+y=10$, where it is evident that the values of x and y may vary through all degrees of *fractional* and *integral* magnitude between 0 and 10; for if $x=\frac{1}{2}$, then $y=9\frac{1}{2}$; if $x=1$, then $y=9$; if $x=1\frac{1}{4}$, then $y=8\frac{3}{4}$; &c. &c.; but if the hypothesis be limited to the *integral and positive* values of x and y, then the number of answers is limited to *nine,* for if $x=1, 2, 3, 4, 5, 6, 7, 8$, or 9, then the corresponding values of y are 9, 8, 7, 6, 5, 4, 3, 2, or 1.

150. Suppose now it was required to find all the integral and positive values of x and y in the equation $2x+3y=17$. Here $x=\frac{17-3y}{2}=8+\frac{1}{2}-y-\frac{y}{2}$; $=8-y-\left(\frac{y-1}{2}\right)$; and since x and y are whole numbers, it is evident that $\frac{y-1}{2}$ must be also a whole number. Let $\frac{y-1}{2}=p$, then $y=2p+1$, and $x=(8-y-p=)8-2p-1-p=7-3p$. To make x a positive number, p cannot be taken greater than 2; let $p=0$, 1, or 2, then $x=7$, 4, or 1, and the corresponding values of $y(=2p+1)$ are 1, 3, and 5; so that the number of positive and integral values of x and y are limited to *three.*

151. Next let it be required to find the same in the equation $14x-5y=7$. Here $y=\frac{14x-7}{5}=\frac{7(2x-1)}{5}$; and since 5 is not a divisor of 7, $\frac{2x-1}{5}$ must be a whole number (Art. 135.)

Let $\frac{2x-1}{5}=p$, then $2x=5p+1$, and $x=2p+\frac{p+1}{2}$;

let $\frac{p+1}{2}=q$, then $p=2q-1$;

$$\text{hence } x=(2p+q=)4q-2+q=5q-2,$$

$$\text{and } y=\frac{7(2x-1)}{5}=\frac{7(10q-5)}{5}=14q-7.$$

$$\left.\begin{array}{l}\text{Let } q=1,\ 2,\ 3,\ 4,\ 5,\ \&c.\\ \text{then } x=3,\ 8,\ 13,\ 18,\ 23,\ \&c.\\ \quad y=7,\ 21,\ 35,\ 49,\ 63,\ \&c.\end{array}\right\}\text{In this case, the positive and integral values of } x \text{ and } y \text{ are } \textit{unlimited}.$$

By attending to the several parts of the process in the two last Articles, the solution of the following Questions will be readily understood.

I. *In how many ways may the sum of £.5 be paid, in crowns and seven-shilling-pieces?* Let x=the N°. of seven-shilling-pieces, y=the N°. of crowns; then $7x+5y=100$, $y=\frac{100-7x}{5}=20-x-\frac{2x}{5}$ (where x must be divisible by 5). Let $\frac{x}{5}=p$, then $x=5p$, and $y=\left(20-x-\frac{2x}{5}=\right)20-5p-2p=20-7p$ (where p must evidently be less than 3). Let $p=1$ or 2, then $x=5$ or 10, and $y=13$ or 6, so that a payment of this sort can only be effected in *two* ways.

II. *What is the least number of pieces in which a bill of £.7 can be paid in half-guineas and seven-shilling-pieces?* Let x=N°. of half-guineas, y=N°. of seven-shilling-pieces, then $21x+14y=280$, or $3x+2y=40$, and $y=\frac{40-3x}{2}=20-x-\frac{x}{2}$ (where x must be divisible by 2). Let $\frac{x}{2}=p$, then $x=2p$, and $y=\frac{40-6p}{2}=20-3p$ (where p must be less than 7).

$$\left.\begin{array}{l}\text{Let } p=1,\ 2,\ 3,\ 4,\ 5,\text{ or } 6,\\ \text{then } x=2,\ 4,\ 6,\ 8,\ 10,\text{ or } 12,\\ \text{and } y=17,\ 14,\ 11,\ 8,\ 5,\text{ or } 2,\end{array}\right\}\text{so that the number of ways in which this payment may be made is } \textit{six}\text{; and the } \textit{least} \text{ N°. of pieces is 14, the } \textit{greatest} \text{ 19.}$$

III. *A person owes me seven shillings; he has no other money about him but half-guineas, and I no other but crown-pieces; what is the least number of pieces by which this debt may be settled?* Let x=N°. of half-guineas, y=N°. of crowns, then $21x-10y=14$, and $y=\frac{21x-14}{10}=2x-1+\frac{x-4}{10}$. Let $\frac{x-4}{10}=p$, then $x=10p+4$, and $y=(20p+8-1+p=)21p+7$ (where p may be 0, or any whole number whatever).

Let

$$\left.\begin{array}{l}\text{Let } p=0,\ 1,\ 2,\ 3,\ 4,\ \&\text{c.}\\ \text{then } x=4, 14, 24, 34, 44, \&\text{c.}\\ \phantom{\text{then }} y=7, 28, 49, 70, 91, \&\text{c.}\end{array}\right\}$$

so that the *least* N°. of pieces is 11, viz 4 half-guineas and 7 crowns; but the *number of ways* in which the payment may be effected is *unlimited.*

IV. *It is required to find the least number which when divided by* 19 *shall leave the remainder* 7; *and when divided by* 28, *the remainder* 13. Let x and y be the *quotients* arising respectively from such division, then $19x+7=28y+13$, and $x=\frac{28y+6}{19}=y+\frac{9y+6}{19}=y+\frac{3(3y+2)}{19}$ (where $3y+2$ must be divisible by 19). Let $\frac{3y+2}{19}=p$, then $y=\frac{19p-2}{3}=6p+\frac{p-2}{3}$; put $\frac{p-2}{3}=q$, then $p=3q+2$; and as it is required to find the *least* number which will answer the conditions required, let $q=0$, then $p=2$, $y=6p$ (for $\frac{p-2}{\ }=0$)$=12$, $x=\frac{28y+6}{19}=18$, in which case $19x+7$ and $28y+13$ are *each* equal to 349, which is the N°. required.

V. *What is the least whole number which, when divided by* 5, 6, 7, *respectively, shall leave remainders* 1, 2, 3? Let x, y, z be the *quotients* arising from this division, then $5x+1=6y+2=7z+3$. Now $x=\frac{6y+1}{5}=y+\frac{y+1}{5}$; let $\frac{y+1}{5}=p$, then $y=5p-1$, and $6y+2=30p-4=7z+3$; hence $z=\frac{30p-7}{7}=4p-1+\frac{2p}{7}$ (where p must be divisible by 7). Let $\frac{p}{7}=q$, then $p=7q$, and $z=\left(4p-1+\frac{2p}{7}=\right)28q-1+2q=30q-1$.

$$\left.\begin{array}{l}\text{Let } q=\ 1,\ \ 2,\ \ 3,\ \ \&\text{c.}\\ \text{then } z=29,\ 59,\ 89,\ \&\text{c.}\\ \text{and } 7z+3=206, 416, 626, \&\text{c.}\end{array}\right\}$$

so that the *least* number which will answer the conditions required is 206.

XLV.

Diophantine Problems.

152. These are a species of unlimited Problems, principally respecting square and cube numbers. No general rules can be laid down for the solution of them; but the following examples may serve to give the learner an insight into their nature, and the manner of solving them.

I. *To find two square numbers whose sum shall also be a square number.* Let x^2 and a^2 represent the two square numbers required; then the values of x^2 and a^2 must be such, that x^2+a^2 may be a square number. Now x^2+a^2 is *greater* than $\overline{x-a}|^2$ (for $\overline{x-a}|^2=x^2+a^2-2ax$); we may therefore assume $x^2+a^2=\overline{mx-a}|^2$, where m is some number *greater than unity*; but if $x^2+a^2=\overline{mx-a}|^2=m^2x^2-2max+a^2$, then $x^2=m^2x^2-2max$, or $m^2x-x=2ma$; $\therefore x=\frac{2ma}{m^2-1}$; hence the two numbers required are $\overline{\frac{2ma}{m^2-1}}|^2$, and a^2, where m and a may be any whole numbers whatever; but that $\frac{2ma}{m^2-1}$ may be an *integer*, it is necessary that $2ma$ be some multiple of m^2-1. Let $m=2$, $a=3$, then the two numbers are 16 and 9, and their sum 25. Let $m=3$, $a=5$, then the two numbers are $\frac{225}{16}$ and 25, whose sum $\frac{625}{16}$ is also a square number. Let $m=3$, $a=8$, then the numbers are 36 and 64, and their sum 100; &c. &c.

II. *To find a number* (x) *such that $x+a$ and $x-a$ shall both be square numbers.* Let $x+a=m^2$, then $x-a=m^2-2a$; assume $m^2-2a=\overline{m-a}|^2=m^2-2ma+a^2$, then $-2a=-2ma+a^2$ or $2ma=a^2+2a$; $\therefore m=\frac{a+2}{2}$, and $m^2=\frac{a^2+4a+4}{4}$; hence $x=m^2-a=\frac{a^2+4a+4}{4}-a=\frac{a^2+4}{4}$ where a may be any number whatever; and if it be an *even* number, then x (and consequently $x+a$ and $x-a$) will be a whole number.

Let $a=1$, then $x=\frac{a^2+4}{4}=\frac{5}{4}$; $x+a=\frac{5}{4}+1=\frac{9}{4}$; $x-a=\frac{5}{4}-1=\frac{1}{4}$,

$a=2$, . . . $x=\frac{4+4}{4}=2$; $x+a=2+2=4$; $x-a=0$,

$a=3$, . . . $x=\frac{9+4}{4}=\frac{13}{4}$; $x+a=\frac{13}{4}+3=\frac{25}{4}$; $x-a=\frac{13}{4}-3=\frac{1}{4}$,

$a=4$, . . . $x=\frac{16+4}{4}=5$, $x+a=5+4=9$; $x-a=5-4=1$;

&c. &c. &c. &c.

and this is a general property of square numbers, viz. that *if we take any number, square it, add 4 to that square, and then divide the result by 4, it will give such a number, that the* sum *and* difference *of it and the* original number *shall be* square numbers.

III. *To find three square numbers which shall be in arithmetic progression.* Let the numbers be x^2, y^2, z^2, then $x^2+z^2=2y^2$. Put $x=p+q$, and $z=p-q$, then $x^2+z^2=2p^2+2q^2=2y^2$, $\therefore p^2+q^2=y^2$, and the question resolves itself into the finding p and q, such that p^4+q^2 shall be a square number. Let, therefore, (Ex. I.) $p=\frac{2ma}{m^2-1}$, $q=a$, then

$$x=p+q=\frac{2ma}{m^2-1}+a$$

$$z=p-q=\frac{2ma}{m^2-1}-a$$

$$y=\sqrt{p^2-q^2}=\frac{a(m^2+1)}{m^2-1}$$

where a and m may be any numbers whatever. For instance, let [illegible]$=3$, $m=2$, then $x=7$, $y=5$, $z=1$, and the square numbers in arithmetic progression are 49, 25, 1. Let $a=8$, $m=3$, then $x=14$, $y=10$, $z=-2$, $\therefore$ the square numbers in arithmetic progression are 196, 100, 4.

XLVI.

The Solution of two Questions relating to Numbers in Geometrical Progression.

153. Let a be the *first term*, r the *common ratio*, n the *number of terms*, and S the *sum* of a Geometric Series; then (by Art. 110), $S=\frac{ar^n-a}{r-1}$; and if $a=1$, $S=\frac{r^n-1}{r-1}$. Now let Σ be the sum of the series arising from the successive addition of 1, 2, 3, 4, &c. . . . n terms of the geometric series; then we shall have,

$$S=1+r+r^2+r^3+r^4+\&c.\ldots\ldots r^{n-1}=\frac{r^n-1}{r-1},\qquad\text{and}$$

$$\Sigma=1+(1+r)+(1+r+r^2)+(1+r+r^2+r^3)+\&c.\ldots+(1+r+r^2+r^3+\&c.\ldots+r^n$$

$$=\frac{r-1}{r-1}+\frac{r^2-1}{r-1}+\frac{r^3-1}{r-1}+\frac{r^4\;\;1}{r-1}+\&c.\ldots\ldots+\frac{r^n-1}{r-1}$$

$$=\frac{1}{r-1}\Big((r-1)+(r^2-1)+(r^3-1)+(r^4-1)+\&c.\ldots+(r^n-1)\Big)$$

$$=\frac{1}{r-1}\Big(r+r^2+r^3+r^4+\&c.\ldots+r^n\Big)-\frac{1}{r-1}\Big(1+1+1+1+\&c.\ldots\text{to } n \text{ terms}\Big)$$

$$=\frac{1}{r-1}\left(\frac{r^{n+1}-r}{r-1}\right)-\frac{n}{r-1}=\frac{r^{n+1}-r}{(r-1)^2}-\frac{n}{r-1};$$

of which the following are Examples.

1. Let $r=2$, then $S=1+2+4+8+16+\&c.\ldots+2^{n-1}=2^n-1$.

$$\Sigma=1+3+7+15+31+\&c.\ldots+2^n-1=2^{n+1}-(n+2).$$

II. Let $r=3$, then $S=1+3+9+27+81+\&c.\ldots+3^{n-1}=\frac{3^n-1}{2}$

$$\Sigma=1+4+13+40+121+\&c.\ldots+\frac{3^n-1}{2}=\frac{3^{n+1}-(2n+3)}{4}.$$

III. Let $r=4$, then $S=1+4+16+64+256+\&c.\ldots+4^{n-1}=\frac{4^n-1}{3}$

$$\Sigma=1+5+21+85+341+\&c.\ldots+\frac{4^n-1}{3}=\frac{4^{n+1}-(3n+1)}{9}$$

&c. &c. &c. = &c.

154. Let $\frac{a}{c}+\frac{a+b}{cr}+\frac{a+2b}{cr^2}+\frac{a+3b}{cr^3}+\frac{a+4b}{cr^4}+$ &c. be an infinite series of fractions whose numerators are in *Arithmetical* and their denominators in *Geometrical* Progression. For finding its sum (S), this series may be resolved into the following;

$$\frac{a}{c}+\frac{a}{cr}+\frac{a}{cr^2}+\frac{a}{cr^3}+\frac{a}{cr^4}+\&c.\ \textit{ad infinitum}=\frac{ar}{c(r-1)}\ ^{(*)}$$

$$\frac{b}{cr}+\frac{b}{cr^2}+\frac{b}{cr^3}+\frac{b}{cr^4}+\&c.\ldots\ldots\ldots=\frac{b}{c(r-1)}$$

$$\frac{b}{cr^2}+\frac{b}{cr^3}+\frac{b}{cr^4}+\&c.\ldots\ldots\ldots=\frac{b}{cr(r-1)}$$

$$\frac{b}{cr^3}+\frac{b}{cr^4}+\&c.\ldots\ldots\ldots=\frac{b}{cr^2(r-1)}$$

$$\frac{b}{cr^4}+\&c.\ldots\ldots\ldots=\frac{b}{cr^3(r-1)}$$

&c. = &c.

Hence

(*) For (by Art. 116) $\frac{a}{c}\left(1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\&c.\right)=\frac{a}{c}\left(\frac{1}{1-\frac{1}{r}}\right)=\frac{ar}{c(r-1)}$.

$$\frac{b}{c}\left(\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\&c.\right)=\frac{b}{c}\left(\frac{\frac{1}{r}}{1-\frac{1}{r}}\right)=\frac{b}{c(r-1)}.$$

$$\frac{b}{c}\left(\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\&c.\right)=\frac{b}{c}\left(\frac{\frac{1}{r^2}}{1-\frac{1}{r}}\right)=\frac{b}{cr(r-1)}.$$

&c. = &c.

Hence $S=\frac{ar}{c(r-1)}\text{[illegible]}\frac{b}{c(r-1)}\left(1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\&c.\ ad\ infinitum\right)$

$=\frac{ar}{c(r-1)}+\frac{b}{c(r-1)}\times\frac{r}{r-1}=\frac{ar}{c(r-1)}+\frac{br}{c(r-1)^2}$; of which the following are examples.

I. Let $a=1$, $b=1$, $c=1$, $r=2$, then

$$S=1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\&c.=2+2=4.$$

II. Let $a=1$, $b=2$, $c=3$, $r=2$, then

$$S=\frac{1}{3}+\frac{3}{6}+\frac{5}{12}+\frac{7}{24}+\frac{9}{48}+\&c.=\frac{2}{3}+\frac{4}{3}=2.$$

III. Let $a=2$, $b=3$, $c=5$, $r=3$, then

$$S=\frac{2}{5}+\frac{5}{15}+\frac{8}{45}+\frac{11}{135}+\frac{11}{405}+\&c.=\frac{6}{10}+\frac{9}{10}=\frac{21}{20}.$$

CHAP. X.

ON THE BINOMIAL THEOREM,

AND SUBJECTS CONNECTED WITH IT.

Sir Isaac Newton's theorem for raising a binomial to any power was given in Chap. III. The index (n) was there supposed to be an *integral* and *positive* number; but the great value and importance of this theorem is derived from its being equally true, whether the index be *integral* or *fractional*, *positive* or *negative*; for this circumstance enables us not only to obtain the roots, as well as powers, of Algebraic quantities in a much more easy manner than by the common processes, but to apply the theorem itself to many very useful and important investigations in the higher branches of analysis.

XLVII.

The general Demonstration of this Theorem.

155. Previously to the investigation of this Theorem, it will be necessary to ascertain the *two first terms* and the *general form* of the series which expresses the value of $(1+ax+bx^2+cx^3+\&c.)^n$, whether n be integral or fractional, positive or negative.[a]

I. If n be a *positive whole number*, then, by the ordinary process of involution exhibited in Art. 49, we have.

$$\begin{array}{l} 1+\ \ ax+\&c. \\ 1+\ \ ax+\&c. \\ \hline 1+\ \ ax+\&c. \\ \ \ \ +\ \ ax+\&c. \\ \hline 1+2ax+\&c. \text{ for the } \textit{Square} \\ 1+\ \ ax+\&c. \\ \hline 1+2ax+\&c. \\ \ \ \ +\ \ ax+\&c. \\ \hline 1+3ax+\&c. \text{ for the } \textit{Cube} \\ 1+\ \ ax+\&c. \\ \hline \ \ \&c.\ \ \ \ \ \ \&c. \end{array}$$

from which it appears, that in finding the value of $(1+ax+bx^2+\&c.)^n$, the *two first terms* will be $1+nax$; and from the nature of the process it is evident that the powers of x will increase regularly.

II. If $n=\frac{1}{r}$, then since the indices of x in the quantity $1+ax+bx^2+cx^3+\&c.$ are all supposed to be *integral* and *positive*, it is evident that the indices of x in the series which expresses the rth *root* of this quantity will be integral and positive also; for if any of the indices in the *root* were fractional or negative, we should, in the re-composition of the power from the root, have fractional or negative indices also in the *power*; which is contrary to the supposition.

With

(a) The general form of a multinomial quantity in which the powers of x regularly ascend is $A+Bx+Cx^2+Dx^3+\&c.$; but this is easily reduced to a form much more simple, yet equally general, by dividing the whole by A; in which case it becomes $1+\frac{B}{A}x+\frac{C}{A}x^2+\frac{D}{A}x^3+\&c.$, or $\left(\text{making } \frac{B}{A}=a,\ \frac{C}{A}=b,\ \frac{D}{A}=c,\ \&c.\right)\ 1+ax+bx^2+cx^3+\&c.$

With respect to the *two first terms* of the root, it is manifest that the first of them will be *unity*, and that the second will be such a quantity as, in the recomposition of the power from the root, will give ax for the second term of the power; now, by Case I., this must be such a quantity as when multiplied by r will produce ax; i.e. it must be $\frac{1}{r}ax$. Hence we have $(1+ax+bx^2+\&c.)^{\frac{1}{r}}=1+\frac{1}{r}ax+\&c.$

$$=1+nax+\&c., \text{ since } \tfrac{1}{r}=n.$$

III. Now let $n=\frac{m}{r}$, then

by involution (Case I.), $(1+ax+bx^2+\&c.)^m=1+max+\&c.$

by extracting the rth root, $(1+ax+bx^2+\&c.)^{\frac{m}{r}}=(1+max+\&c.)^{\frac{1}{r}}$

$$=1+\frac{1}{r}(max)+\&c. \text{(by Case II.)}$$

$$=1+\frac{m}{r}(ax)+\&c.$$

$$=1+nax+\&c. \text{ as in Cases I. II.}$$

IV. If $n=-s$, where s is either integral or fractional, then $(1+ax+bx^2+\&c.)^{-s}=\frac{1}{(1+ax+bx^2+\&c.)^s}$

$$=\frac{1}{1+sax+\&c.} \text{(by Cases I. II. III.)}$$

$$=1-sax+\&c. \text{ by actual division.}$$

$$=1+nax+\&c. \text{ as in former cases.}$$

Hence it appears, that whether n be *integral* or *fractional*, *positive* or *negative*, the first two terms of the series expressing the value of $(1+ax+bx^2+\&c)^n$ will be $1+nax$, and that in the subsequent terms the powers of x will be integral and positive.

Now, suppose $a=1$; $b=0$; $c=0$; &c. then the multinomial quantity $1+ax+bx^2+\&c.$ is reduced to the binomial $1+x$; and we are evidently at liberty to assume

$$(1+x)^n=1+nx+qx^2+rx^3+sx^4+\&c.$$

where $q, r, s,$ &c. are quantities whose values are hereafter to be determined. Hence, also, since $(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$,

we have $(a+x)^n = a^n\left(1+\frac{nx}{a}+\frac{qx^2}{a^2}+\frac{rx^3}{a^3}+\&c.\right)$

$$= a^n + na^{n-1}x + qa^{n-2}x^2 + ra^{n-3}x^3 + \&c.$$

156. Now let the trinomial quantity $(1+x+h)^n$ be expanded, first by considering $x+h$ as *one quantity*, and secondly by considering $1+x$ as *one quantity*, and there will arise two series, from the comparison of which the values of q, r, s, &c. may be obtained. Thus

I. $\left(1+(x+h)\right)^n = 1+n(x+h)+q(x+h)^2+r(x+h)^3+s(x+h)^4+\&c$

$$= 1+nx+qx^2+rx^3+sx^4+\&c.+nh+2qhx+3rhx^2+4shx^3+\&c. (A)$$

omitting the higher powers of h, as unnecessary for our purpose.

II. $\left((1+x)+h\right)^n = (1+x)^n + n(1+x)^{n-1}h + \&c$

$$= 1+nx+qx^2+rx^3+sx^4+\&c.+nh\left(1+(n-1)x+q'x^2+r'x^3+s'x^4+\&c\right)^{(a)}$$

$$= 1+nx+qx^2+rx^3+sx^4+\&c.+nh+n(n-1)hx+nq'hx^2+nr'hx^3+\&c (B)$$

Since the series (A) and (B) arise from the expansion of the same quantity $1+x+h$, they are evidently *equal*; rejecting therefore the part common to both, we have

$$2qhx+3rhx^2+4shx^3+\&c. = n(n-1)hx+nq'hx^2+nr'hx^3+\&c.$$

and equating the coefficients(b), we have

$2q = n(n-1)$, or $q = \frac{n(n-1)}{2}$; and by parity of reason, $q' = \frac{(n-1)(n-2)}{2}$(c)

$3r = nq' = \frac{n(n-1)(n-2)}{2}$, or $r = \frac{n(n-1)(n-2)}{2.3}$; . . . . . . $r' = \frac{(n-1)(n-2)(n-3)}{2.3}$

$4s = nr' = \frac{n(n-1)(n-2)(n-3)}{2.3}$, or $s = \frac{n(n-1)(n-2)(n-3)}{2.3.4}$; $s' = \frac{(n-1)(n-2)(n-3)(n-4)}{2.3.4}$

&c. = &c = &c..

By

(a) In assuming a series for the value of $(1+x)^{n-1}$, the first two terms (by Art. 155) will be $1+(n-1)x$; and the other coefficients will also be different from those of the series which expresses the value of $(1+x)^n$. To preserve an uniformity of notation, we have made them q', r', s', &c.

(b) This process of equating coefficients requires explanation; for which purpose, let us suppose $a+bx+cx^2+dx^3+\&c.$ and $\alpha+\beta x+\gamma x^2+\delta x^3+\&c.$ to be two series arising from the different modes of expansion (by *division*, *evolution*, &c.) of the same quantity, or of equal quantities, in which a, b, c, d, &c. and α, β, γ, δ, &c. are *constant* quantities, but x a quantity varying through all degrees of magnitude. Since the two series

are

By substituting the values of p, q, r, &c. thus found, we have

$$(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{2.3}x^3+\frac{n(n-1)(n-2)(n-3)}{2.3.4}x^4+\text{\&c. and}$$

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}x^3+\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}$$

so that the series expressing the value of $(a+x)^n$, n being any number whatever, either *integral* or *fractional*, *positive* or *negative*, observes the same law as that which was deduced in Chap. III., on the supposition of n being a *positive whole number*.

XLVIII.

Some Observations arising out of the foregoing Theorem.

157. Resuming the notation adopted in Chap. III. we have $(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+$ &c. the m^{th} term of the series being $\frac{n(n-1)(n-2)\dots(n-m+2)}{1.2.3\dots(m-1)}a^{n-m+1}b^{m-1}$. Hence, if n be a *positive whole number*, the series

---

are equal to one another, whatever be the value of x, let us suppose $x=0$, and we have $a=\alpha$; and these two quantities being invariable, a will always be equal to α for *every* value of x. Now since $a=\alpha$, $bx+cx^2+dx^3+$&c. must be equal to $\beta x+\gamma x^2+\delta x^3+$&c.; divide by x, and we have $b+cx+dx^2+$&c.$=\beta+\gamma x+\delta x^2+$&c.; suppose again $x=0$, then $b=\beta$, and so on; hence $a=\alpha$, $b=\beta$, $c=\gamma$, $d=\delta$, &c. The same is also true in the equation $(a+bx+cx^2+\text{\&c.})y+Py^2+Qy^3+\text{\&c.}=(\alpha+\beta x+\gamma x^2+\text{\&c.})y+P'y^2+Q'y^3+$&c.; for divide by y, then $a+bx+cx^2+$&c.$+Py+Qy^2+$&c.$=\alpha+\beta x+\gamma x^2+$&c.$+P'y+Q'y^2+$&c.; let $y=0$, then $a+bx+cx^2+$&c.$=\alpha+\beta x+\gamma x^2+$&c. and a, b, c, &c. may be proved equal to α, β, γ, &c. respectively, as before.

(c) For if the coefficient of the third term of the series which expresses the value of $(1+x)^n$ be $\frac{n(n-1)}{2}$, the coefficient of the third term of the series which expresses the value of $(1+x)^{n-1}$ will (by substituting $n-1$ for n) be $\frac{(n-1)(n-2)}{2}$; and so of the rest, r', s', &c.

series will *terminate* after $n+1$ terms; for let $m=n+2$, then $n-m+2=0$, and consequently the coefficient which involves the factor $(n-m+2)$ *vanishes*. Let $m=n+1$, then $n-m+2=1$, $n-m+1=0$, and $m-1=n$; ∴ the $(n+1$th$)$ (or *last*) term is $\frac{n(n-1)(n-2)....3.2.1}{1.2.3....(n-1)n}$ a^0b^n or b^n. If n be *fractional* or *negative*, the series will not terminate, and in this case the value of any expanded binomial can only be expressed in the form of an *infinite series*.

158. If in the series expressing the value of $(a+b)^n$, for b we put $-b$, then those terms which involve the *odd* powers of b will be changed from + to − ; Hence we have,

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\&c.$$

$$\text{and } (a-b)^n=a^n-na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\&c.$$

∴ by *addition*, $(a+b)^n+(a-b)^n=2a^n+n(n-1)a^{n-2}b^2+\&c.$

or $\frac{1}{2}(a+b^n)+\frac{1}{2}(a-b)^n=a^n+\frac{n(n-1)}{2}a^{n-2}b^2+\&c.$

by *subtraction*, $(a+b)^n-(a-b)^n=2na^{n-1}b+\frac{n(n-1)(n-2)}{3}a^{n-3}b^3+\&c.$

or $\frac{1}{2}(a+b)^n-\frac{1}{2}(a-b)^n=na^{n-1}b+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\&c.$

159. Let $a=1$, $b=1$, then $(a+b)^n=(1+1)^n=2^n$; and since the several powers of a and b are, in this case, each of them equal to 1, we have $1+n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{2.3}+$ &c.$=2^n$, i.e. *the sum of the coefficients of the nth power of any binomial is equal to that power of 2 whose index is* n. Thus, for the *square*, $1+2+1=4=2^2$; for the *cube* $1+3+3+1=8=2^3$; for the *fourth* power, $1+4+6+4+1=16=2^4$; &c. &c. If $a=1$, $b=1$, in the expression $(a-b)^n$, then $(1-1)^n=0$, which shews that the sum of the *positive* coefficients of $(a-b)^n$ is equal to the sum of the *negative* ones.

XLIX.

On the Expansion of Series.

160. It has already been observed (Art. 157) that when n is a *negative* number or a *fraction*, then the series expressing the value of $(a+b)^n$ does not terminate. Let $n=\frac{m}{r}$, and substitute $\frac{m}{r}$ for n in the series (Art. 156); then

$$(a+b)^{\frac{m}{r}}=a^{\frac{m}{r}}+\frac{m}{r}a^{\frac{m}{r}-1}b+\frac{\frac{m}{r}\left(\frac{m}{r}-1\right)}{2}a^{\frac{m}{r}-2}b^2+\frac{\frac{m}{r}\left(\frac{m}{r}-1\right)\left(\frac{m}{r}-2\right)}{2.3}a^{\frac{m}{r}-3}b^3+\&c.$$

$$=a^{\frac{m}{r}}+\frac{m\,a^{\frac{m}{r}}}{r}\left(\frac{b}{a}\right)+\frac{m(m-r)a^{\frac{m}{r}}}{2r^2}\left(\frac{b^2}{a^2}\right)+\frac{m(m-r)(m-2r)a^{\frac{m}{r}}}{2.3.r^3}\left(\frac{b^3}{a^3}\right)+\&c.^{(a)}$$

$$=a^{\frac{m}{r}}\Bigg(1+\frac{m}{r}\left(\frac{b}{a}\right)+\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)+\frac{m(m-r)(m-2r)}{2.3.r^3}\left(\frac{b^3}{a^3}\right)+\&c.$$

which is a general expression for finding the value of any binomial surd quantity in a series, $\frac{m}{r}$ being either positive or negative, and m and r any whole numbers whatever.

EXAMPLE 1.

Find the value of $\sqrt[3]{c^3+x^3}$ or $\overline{c^3+x^3}|^{\frac{1}{3}}$ in a series.

$$\text{Here }\left.\begin{aligned}a&=c^3\\ b&=x^3\\ m&=1\\ r&=3\end{aligned}\right\}\qquad \begin{aligned}&\therefore a^{\frac{m}{r}}=\sqrt[3]{c^3}=c;\\ &\frac{m}{r}\left(\frac{b}{a}\right)=\frac{1}{3}\left(\frac{x^3}{c^3}\right)=\frac{x^3}{3c^3};\end{aligned}$$

(a) This series is derived from the preceding one, by resolving the powers of a into two factors; thus, $a^{\frac{m}{r}-1}=a^{\frac{m}{r}}\times a^{-1}=a^{\frac{m}{r}}\times\frac{1}{a}=\frac{a^{\frac{m}{r}}}{a}$; $a^{\frac{m}{r}-2}=a^{\frac{m}{r}}\times a^{-2}=a^{r}\times\frac{1}{a^2}=\frac{a^{\frac{m}{r}}}{a^2}$.

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{1(1-3)}{2.3^2}\left(\frac{x^6}{c^6}\right)=-\frac{x^6}{3^2.c^6};$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{1(1-3)(1-6)}{2.3.3^3}\left(\frac{x^9}{c^9}\right)=\frac{5x^9}{3^4.c^9};$$

&c. = &c.

Hence $\sqrt[3]{c^3+x^3}=c\left(1+\frac{x^3}{3c^3}-\frac{x^6}{3^2.c^6}+\frac{5x^9}{3^4.c^9}-\text{\&c.}\right)$

Ex. 2.

Find the value of $\frac{d}{\sqrt{c^2+x^2}}$ or $\frac{d}{(c^2+x^2)^{\frac{1}{2}}}$ or $d(c^2+x^2)^{-\frac{1}{2}}$.

Here $a=c^2$ $\quad\therefore a^{\frac{m}{r}}=(c^2)^{-\frac{1}{2}}=c^{-1}=\frac{1}{c}$;

$b=x^2$

$m=-1$ $\quad\frac{m}{r}\left(\frac{b}{a}\right)=-\frac{1}{2}\left(\frac{x^2}{c^2}\right)=-\frac{x^2}{2c^2}$;

$r=2$

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{-1(-1-2)}{2.2^2}\left(\frac{x^4}{c^4}\right)=\frac{3x^4}{2^3.c^4};$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{-1(-1-2)(-1-1)}{2.3.2^3}\left(\frac{x^6}{c^6}\right)=-\frac{5x^6}{2^4.c^6};$$

Hence $(c^2+x^2)^{-\frac{1}{2}}=\frac{1}{c}\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2^3.c^4}-\frac{5x^6}{2^4.c^6}+\text{\&c.}\right)$

and $\frac{d}{\sqrt{c^2+x^2}}=\frac{d}{c}\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2^3.c^4}-\frac{5x^6}{2^4.c^6}+\text{\&c.}\right)$

Ex. 3.

Find the value of $\frac{1}{(c+x)^2}$, or $(c+x)^{-2}$.

Here $a=c$ $\quad\therefore a^{\frac{m}{r}}=c^{-2}=\frac{1}{c^2}$;

$b=x$

$m=-2$ $\quad\frac{m}{r}\left(\frac{b}{a}\right)=-\frac{2x}{c}$;

$r=1$

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{-2(-2-1)}{2}\left(\frac{x^2}{c^2}\right)=\frac{3x^2}{c^2};$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{-2(-2-1)(-2-2)}{2.3}\left(\frac{x^3}{c^3}\right)=-\frac{4x^3}{c^3};$$

&c. = &c.

Hence $\frac{1}{(c+x)^2}=\frac{1}{c^2}\left(1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+\&c.\right)$

This series is easily verified by the division of 1 by $c^2+2cx+x^2$.

Ex. 4.

Find the value of $(c^2-x^2)^{\frac{3}{4}}$.

Here $\left.\begin{array}{l} a=c^2 \\ b=-x^2 \\ m=3 \\ r=4 \end{array}\right\}$ $\begin{array}{l} \therefore a^{\frac{m}{r}}=\sqrt[4]{c^6}=\sqrt{c^3}; \\ \frac{m}{r}\left(\frac{b}{a}\right)=\frac{3}{4}\left(-\frac{x^2}{c^2}\right)=-\frac{3x^2}{2^2.c^2}; \end{array}$

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{3(3-4)}{2.4^2}\left(\frac{x^4}{c^4}\right)=-\frac{3x^4}{2^5.c^4};$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{3(3-4)(3-8)}{2.3.4^3}\left(-\frac{x^6}{c^6}\right)=-\frac{5x^6}{2^7.c^6};$$

&c.=&c.

Hence $(c^2-x^2)^{\frac{3}{4}}=\sqrt{c^3}\left(1-\frac{3x^2}{2^2.c^2}-\frac{3x^4}{2^5.c^4}-\frac{5x^6}{2^7.c^6}-\&c.\right)$

161. Now let $m=1$, then $(a+b)^{\frac{m}{r}}=(a+b)^{\frac{1}{r}}=\sqrt[r]{a+b}$; and $a^{\frac{m}{r}}=\sqrt[r]{a}$; hence the series in Art. 160 is transformed into $\sqrt[r]{a+b}=\sqrt[r]{a}\left(1+\frac{1}{r}\left(\frac{b}{a}\right)+\frac{1-r}{2r^2}\left(\frac{b^2}{a^2}\right)+\frac{(1-r)(1-2r)}{2.3.r^3}\left(\frac{b^3}{a^3}\right)+\frac{(1-r)(1-2r)(1-3r)}{2.3.4r^4}\left(\frac{b^4}{a^4}\right)+\&c.\right)(A)$.

Let $a=1$, $b=1$, then

$\sqrt[r]{2}=1+\frac{1}{r}+\frac{1-r}{2r^2}+\frac{(1-r)(1-2r)}{2.3r^3}+\frac{(1-r)(1-2r)(1-3r)}{2.3.4r^4}$ $+\&c.$ (B). Thus

If $r=2$, then $\sqrt{2}=1+\frac{1}{2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{5}{2^7}+\frac{7}{2^8}-\frac{3.7}{2^{10}}+\&c.$

$r=3$, . . . $\sqrt[3]{2}=1+\frac{1}{3}-\frac{1}{3^2}+\frac{5}{3^4}-\frac{2.5}{3^5}+\frac{2.11}{3^6}-\frac{2.7.11}{3^8}+\&c.$

&c.=&c.

By means of the series marked (A), the rth root of many other numbers may be found, if a and b be so assumed, that

b is a small number with respect to a, and $\sqrt[r]{a}$ a whole number; thus,

Example 1.

Let $a=4$, $b=1$, $r=2$, then $\sqrt[r]{a}=\sqrt{4}=2$, and we have

$$\sqrt{4+1}=\sqrt{5}=2\left(1+\frac{1}{2^3}-\frac{1}{2^7}+\frac{1}{2^{10}}-\frac{5}{2^{15}}+\&c.\right)$$

Ex. 2.

Let $a=8, b=1$, $r=3$, then $\sqrt[r]{a}=\sqrt[3]{8}=2$, and we obtain

$$\sqrt[3]{8+1}=\sqrt[3]{9}=2\left(1+\frac{1}{3.8}-\frac{1}{3^2.8^2}+\frac{5}{3^4.8^3}-\frac{2.5}{3^5.8^4}+\&c.\right)$$

Ex. 3.

Let $a=8, b=-2, r=3$, then $\frac{b}{a}=-\frac{2}{8}=-\frac{1}{4}$, and we have

$$\sqrt[3]{8-2}=\sqrt[3]{6}=2\left(1-\frac{1}{3.4}-\frac{1}{3^2.4^2}-\frac{5}{3^4.4^3}-\frac{2.5}{3^5.4^4}-\&c.\right)$$

The several terms of these series are found by substituting for a, b, and r their values in the general series marked (A) or (B), and then rejecting the factors common to both the numerators and denominators of the fractions. Thus, for instance, to find the *seventh* term of the series exhibiting the value of $\sqrt{2}$, we take the 7th term of the series marked B, which is $\frac{(1-r)(1-2r)(1-3r)(1-4r)(1-5r)}{2.3.4.5.6r^6}$; and since $r=2$, the fraction is $-\frac{3.5.7.9}{2.3.4.5.6.2^6}=-\frac{7.9}{2.4.6.2^6}=\left(\text{since } \frac{9}{6}=\frac{3}{2}\right)-\frac{3.7}{2.4.2.2^6}=-\frac{3.7}{2^{10}}$. To find the 5th term of the series expressing the proximate value of $\sqrt[3]{9}$, we take the 5th term of the general series marked (A), which is $\frac{(1-r)(1-2r)(1-3r)}{2.3.4.r^4}\left(\frac{b^4}{a^4}\right)$, where $a=8$, $b=1$, and $r=3$;

$\therefore$ the value of the fraction is $-\frac{2.5.8}{2.3.4.3^4}\left(\frac{1}{8^4}\right)=-\frac{2.5}{3.3^4.8^4}$ $=-\frac{2.5}{3^5.8^4}$. In this manner each term of the several series is calculated.

162. These series converge very fast, so that a few terms would give the rth root of certain numbers with a great degree of accuracy. But a more practical method of finding the higher roots of such numbers, is, by making the number whose root is to be extracted equal to $a^r + b$, and then assuming $a + x = \sqrt[r]{a^r + b}$, x being some decimal fraction; for in this case $(a + x)^r = a^r + b$, and by expanding $(a + x)^r$ and neglecting all the powers of x after x^2 (being very small compared with the preceding ones) we have

$$a^r + ra^{r-1}x + r\left(\frac{r-1}{2}\right)a^{r-2}x^2 = a^r + b;$$

$$\therefore ra^{r-1}x + r\left(\frac{r-1}{2}\right)a^{r-2}x^2 = b \ (A)$$ an equation from which the value of x may be found in two ways.

I. By arranging the terms, and dividing by $r\left(\frac{r-1}{2}\right)a^{r-2}$, we have $x^2 + \frac{2ax}{r-1} = \frac{2b}{r(r-1)a^{r-2}}$;

$$\therefore x^2 + \frac{2ax}{r-1} + \frac{a^2}{(r-1)^2} = \frac{2b}{r(r-1)a^{r-2}} + \frac{a^2}{(r-1)^2};$$

and by solving the quadratic,

$$x = -\frac{a}{r-1} + \sqrt{\frac{2b}{r(r-1)a^{r-2}} + \frac{a^2}{(r-1)^2}}.$$

Hence $\sqrt[r]{a^r + b} = a + x = \frac{r-2}{r-1}a + \sqrt{\frac{2b}{r(r-1)a^{r-2}} + \frac{a^2}{(r-1)^2}}$, which is **Halley**'s Rule, (*Philosophical Transactions*, 1694.)

II. From equation (A) we have $x\left(ra^{r-1} + r\left(\frac{r-1}{2}\right)a^{r-2}x\right) = b$,

$$\therefore x = \frac{b}{ra^{r-1} + r\left(\frac{r-1}{2}\right)a^{r-2}x} = \frac{b}{ra^{r-2}}\left(\frac{1}{a + \frac{r-1}{2}x}\right).$$

By a *first* approximation, neglecting the term which involves x, we have $x = \frac{b}{ra^{r-1}}$; substitute this value for x in the fraction $\frac{b}{ra^{r-2}}\left(\frac{1}{a + \frac{r-1}{2}x}\right)$, and we obtain a *second* approximation, which gives $x=\frac{\text{[illegible]}}{ra^{r-1}}\left(\frac{1}{a+\frac{r-1}{2r}\left(\frac{b}{a^{r-1}}\right)}\right)$

$$\text{and}\quad \sqrt[r]{a^r+b}=a+x=a+\frac{b}{ra^{r-1}}\left(\frac{1}{a+\frac{r-1}{2r}\left(\frac{b}{a^{r-1}}\right)}\right)$$

which is the Rule given by La Croix (*Complément d'Algèbre*), and ascribed to Lambert.

Example 1.

Find an approximate value of the cube root of 67.

Here $67=64+3=4^3+3$; $\therefore a=4$, $b=3$, $r=3$; hence, by the *first* method, $a+x=\frac{1}{2}a+\sqrt{\frac{b}{3a}+\frac{a^2}{4}}$, or $\sqrt[3]{67}=2+\sqrt{\frac{1}{4}+4}=2+2.0615=4.0615$.

Ex. 2.

Find an approximate value of the fifth root of 30.

Here $30=32-2=2^5-2$; $\therefore a=2$, $b=-2$, $r=5$; hence, by the *second* method, $a+x=a+\frac{b}{5a^4}\left(\frac{1}{a+\frac{4b}{10a^4}}\right)$, or

$$\sqrt[5]{30}=2-\frac{1}{20}\left(\frac{1}{2-\frac{1}{20}}\right)=2-\frac{1}{39}=\frac{77}{39}=1.9743.$$

The method of finding the *r*th root of certain numbers, as exhibited in this and the foregoing Article, is a matter rather of curiosity than practical utility, as the *r*th root of any number whatever may be found with great facility by means of *Logarithms*. This method would be useful, however, in an operation where it was required to express this root in the form of a *vulgar fraction*; as in the last Example, where we obtained the approximate value of the 5th root of 30 in the shape of the fraction $\frac{77}{39}$.

L.

On the method of finding the approximate Ratio of the Powers and Roots of Numbers whose Difference is small.

163. Let $a+x$ and a be two numbers whose difference is x, then $(a+x)^n : a^n :: a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2.3}a^{n-3}x^3 + \&c. : a^n$ :: (dividing each term of the ratio by a^{n-1}) $a + nx + \frac{n(n-1)}{2}\left(\frac{x^2}{a}\right) + \frac{n(n-1)(n-2)}{2.3}\left(\frac{x^3}{a^2}\right) + \&c. : a$.

164. Suppose now that n is not a large number, and that x is very small when compared with a, then the fractions $\frac{x^2}{a}$, $\frac{x^3}{a^2}$, &c. will be small also, and those terms in which they are involved will be very small when compared with the *integral* part $a+nx$ of the series; in this case, therefore, the ratio of $(a+x)^n : a^n$ approximates to the ratio of $a+nx : a$. Thus the ratio of $(a+x)^2 : a^2$ approximates to the ratio of $a+2x : a$; of $(a+x)^3 : a^3$ to the ratio of $a+3x : a$; &c. &c.; or if $n=\frac{1}{2}, \frac{1}{3}$, &c. then the ratio of $\sqrt{a+x} : \sqrt{a}$ approximates to the ratio of $a+\frac{1}{2}x : a$; of $\sqrt[3]{a+x} : \sqrt[3]{a}$ to the ratio of $a+\frac{1}{3}x : a$; &c. &c. For instance, the ratio of the *square* of 501 to the square of 500 (in which case, $a=500$, $x=1$, $n=2$) is 502 : 500 very nearly; the ratio of the *cube* of 62 to the cube of 61, is 64 : 61 very nearly; &c. &c. Again, the ratio of the *square root of* 501 to the square root of 500 is $500\frac{1}{2}$: 500; and of the *cube root* of 103 to the cube root of 100, is 101 : 100, very nearly.

165. If the difference between the two numbers is not very small when compared with the numbers themselves, then the *three* first terms of the series must be taken instead of *two,* in which case the approximate ratio of $(a+x)^n : a^n$ becomes that of $a + nx + \frac{n(n-1)}{2}\left(\frac{x^2}{a}\right) : a$. For instance, let

it be required to find a near approximation to the ratio of $\sqrt[3]{11} : \sqrt[3]{10}$, then $a=10$, $x=1$, $n=\frac{1}{3}$, and the approximate ratio becomes that of $10+\frac{1}{3}-\frac{1}{90} : 10$, or of $\frac{900+30-1}{90} : 10$, or of $929 : 900$. By the Theorem in Art. 164, this approximation would be $10\frac{1}{3} : 10$, or $31 : 30$, i.e. $930 : 900$.

Another method, which gives a much nearer approximation, is as follows. Let $S=$ half the sum of the given numbers, and $D=$ half their difference; then (Art. 28) the numbers themselves will be $S+D$ and $S-D$. Hence the ratio of their nth powers is that of $S^n+nS^{n-1}D+$ &c. : $S^n-nS^{n-1}D+$ &c. or of $S+nD+$ &c. : $S-nD+$ &c. and their *approximate* ratio that of $S+nD : S-nD$. If this method be applied to the last Example, $S=\frac{21}{2}$, $D=\frac{1}{2}$, and the approximate ratio is that of $\frac{21}{2}+\frac{1}{6} : \frac{21}{2}-\frac{1}{6}$, or of $64 : 62$, or of $32 : 31$, which is nearer the truth than that of $929 : 900$, given by the last method.

CI.

On the method of extracting the n*th Root of a Binomial Quadratic Surd.*

166. In the expression $x+\sqrt{y}$, let x be a rational quantity and $\sqrt{y}$ a quadratic surd, then $(x+\sqrt{y})^n = x^n + n\,x^{n-1}\sqrt{y} + n\,.\,\frac{n-1}{2}x^{n-2}y + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}x^{n-3}y\sqrt{y} +$ &c. (P). Let the sum of the *rational* terms in the series (P) be equal to a, and of the *irrational* $=p\sqrt{y}=\sqrt{p^2y}$, which may be expressed in the form $\sqrt{b}$, $\sqrt{b}$ being a quadratic surd containing the surd $\sqrt{y}$. Hence $(x+\sqrt{y})^n = a+\sqrt{b}$, and $\sqrt[n]{a+\sqrt{b}} = x+\sqrt{y}$; if, therefore, the nth root of a quadratic surd of the form $a+\sqrt{b}$ can be extracted, it *may* be expressed under the form $x+\sqrt{y}$, whether n be an *odd* or *even* number.

167. Let $\sqrt{x}+\sqrt{y}$ be a binomial quadratic surd, in which $\sqrt{x}$ and $\sqrt{y}$ are surds not reducible to the same irrational part, then $(\sqrt{x}+\sqrt{y})^n=$

$$x^{\frac{n}{2}}+nx^{\frac{n-1}{2}}\sqrt{y}+n.\frac{n-1}{2}x^{\frac{n-2}{2}}y+n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y}+\&c.(Q).$$

If n be an *even* number, then the 1st, 3d, 5th, &c. terms of the series (Q) are *rational,* and the 2d, 4th, 6th, &c. *irrational,* (Art. 129.) and $(\sqrt{x}+\sqrt{y})^n$ may (as before) be expressed under the form $a+\sqrt{b}$, where $\sqrt{b}$ is a quadratic surd containing the surd $\sqrt{xy}$. Hence $(\sqrt{x}+\sqrt{y})^n=a+\sqrt{b}$, or $\sqrt[n]{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$; and, if the nth root of $a+\sqrt{b}$ can be extracted, it *may* be expressed under the form $\sqrt{x}+\sqrt{y}$.

168. If n be an *odd* number, then $\frac{n}{2}, \frac{n-2}{2}$, &c. are *fractions,* and $\frac{n-1}{2}, \frac{n-3}{2}$, &c. *whole numbers;* hence the 1st, 3d, 5th, &c. terms of the series (Q) will be surd quantities involving $\sqrt{x}$, and the 2d, 4th, 6th, &c. terms, surd quantities involving $\sqrt{y}$; the series may therefore be expressed under the form $p\sqrt{x}+q\sqrt{y}=\sqrt{p^2x}+\sqrt{q^2y}$, or under the more general form $\sqrt{a}+\sqrt{b}$, where $\sqrt{a}$ and $\sqrt{b}$ are quadratic surds involving the surds $\sqrt{x}$ and $\sqrt{y}$ respectively. In this case, then, $(\sqrt{x}+\sqrt{y})^n=\sqrt{a}+\sqrt{b}$, or $\sqrt[n]{\sqrt{a}+\sqrt{b}}=\sqrt{x}+\sqrt{y}$; and consequently, if the nth root of $\sqrt{a}+\sqrt{b}$ can be extracted, it *may* be expressed under the form $\sqrt{x}+\sqrt{y}$.

169. From hence it appears that the nth root of $a+\sqrt{b}$ may be expressed under the form $x+\sqrt{y}$, whether n be an *odd* or an *even* number; that the nth root of $a+\sqrt{b}$ may *also* be expressed under the form $\sqrt{x}+\sqrt{y}$, when n is an *even* number; but that the nth root of $\sqrt{a}+\sqrt{b}$ can be expressed in the form of a binomial quadratic surd only when n is an odd number, and then under the form $\sqrt{x}+\sqrt{y}$.

170.* Suppose now that $\sqrt[n]{a+\sqrt{b}}=x+\sqrt{y}$, then $a+\sqrt{b}=(x+\sqrt{y})^n=x^n+nx^{n-1}\sqrt{y}+n.\frac{n-1}{2}x^{n-2}y+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}.y\sqrt{y}+\&c.$; ∴ by Art. 132, $a=x^n+n.\frac{n-1}{2}x^{n-2}y+\&c.$ and $\sqrt{b}=nx^{n-1}\sqrt{y}+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y\sqrt{y}+\&c.$; hence $a-\sqrt{b}=x^n-nx^{n-1}\sqrt{y}+n.\frac{n-1}{2}x^{n-2}y-n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y\sqrt{y}+\&c.=(x-\sqrt{y})^n$, or $\sqrt[n]{a-\sqrt{b}}=x-\sqrt{y}$; from which it appears that if $\sqrt[n]{a+\sqrt{b}}=x+\sqrt{y}$, then will $\sqrt[n]{a-\sqrt{b}}=x-\sqrt{y}$.

In the same manner, if $\sqrt[n]{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, where n is an *even* number, it may be shewn that $\sqrt[n]{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

171. Let $\sqrt[n]{\sqrt{a}+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ (n being an *odd* number), then $\sqrt{a}+\sqrt{b}=(\sqrt{x}+\sqrt{y})^n=x^{\frac{n}{2}}+nx^{\frac{n-1}{2}}\sqrt{y}+n.\frac{n-1}{2}x^{\frac{n-2}{2}}y+n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y}+\&c.$; hence by Art. 132, (since $\sqrt{a}$ is a quadratic surd involving $\sqrt{x}$, and $\sqrt{b}$ a quadratic surd involving $\sqrt{y}$) $\sqrt{a}=x^{\frac{n}{2}}+n.\frac{n-1}{2}x^{\frac{n-2}{2}}y+\&c.$ and $\sqrt{b}=nx^{\frac{n-1}{2}}\sqrt{y}+n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y}+\&c.$; ∴ $\sqrt{a}-\sqrt{b}=x^{\frac{n}{2}}-nx^{\frac{n-1}{2}}\sqrt{y}+n.\frac{n-1}{2}x^{\frac{n-2}{2}}y-n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y}+\&c.=(\sqrt{x}-\sqrt{y})^n$ or $\sqrt[n]{\sqrt{a}-\sqrt{b}}=\sqrt{x}-\sqrt{y}$; from which it follows, that if $\sqrt[n]{\sqrt{a}+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ then $\sqrt[n]{\sqrt{a}-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

172. Suppose now that $A+B$ is a binomial quadratic surd, one or both of whose terms is irrational, then, from what has been shewn, it appears that if A and B are both irrational, the nth root of $A+B$ can be extracted only when n is an *odd* number; but if A be *rational*, then the nth root of $A+B$ *may* be extracted, whether n be an *odd* or an *even*

number. In the following Rule for extracting the nth root of $A+B$, the two terms are supposed to be so arranged that A is greater than B; and it consists of two cases, depending upon the value of A^2-B^2.

Case I.

When A^2-B^2 is a complete nth power; i. e. when $A^2-B^2=\alpha^n$, or $\sqrt[n]{A^2-B^2}=\alpha$, α being some whole number.

$$\text{Assume } \sqrt[n]{A+B}=\sqrt{x}+\sqrt{y}\,(R),$$

then (by Art. 170 or 171.) $\sqrt[n]{A-B}=\sqrt{x}-\sqrt{y}\,(S)$;

$$\therefore \sqrt[n]{A^2-B^2}\,(\alpha)=x-y.$$

By squaring equation (R), $\sqrt[n]{A^2+B^2+2AB}=x+y+2\sqrt{xy}$,

. . . . equation (S), $\sqrt[n]{A^2+B^2-2AB}=x+y-2\sqrt{xy}$;

$\therefore \sqrt[n]{A^2+B^2+2AB}+\sqrt[n]{A^2+B^2-2AB}=2x+2y=$ some *whole number.*

Now let $\sqrt[n]{A^2+B^2+2AB}=p+f$, where p is the nearest whole number *less* than the true root, and consequently f a proper fraction;

and let $\sqrt[n]{A^2+B^2-2AB}=q-f'$, where q is the nearest whole number *greater* than the true root, and consequently f' a proper fraction.

Then $\sqrt[n]{A^2+B^2+2AB}+\sqrt[n]{A^2+B^2-2AB}=p+q+f-f'$ $=2x+2y=$ a whole number,

$\therefore f-f'=0,$[a] or $f=f'$; hence $\sqrt[n]{A^2+B^2+2AB}+\sqrt[n]{A^2+B^2-2AB}=p+q$; let $p+q=t$, then

$$\left.\begin{array}{r}2x+2y=t \text{ or } x+y=\tfrac{1}{2}t\\ \text{but } x-y=\alpha\end{array}\right\}\therefore \begin{array}{l}2x=\tfrac{1}{2}t+\alpha, \text{ and } \sqrt{x}=\tfrac{1}{2}\sqrt{t+2\alpha},^{(b)}\\ 2y=\tfrac{1}{2}t-\alpha, \text{ and } \sqrt{y}=\tfrac{1}{2}\sqrt{t-2\alpha}.\end{array}$$

$$\text{Hence } \sqrt[n]{A\pm B}=\sqrt{x}\pm\sqrt{y}=\tfrac{1}{2}\sqrt{t+2\alpha}\pm\tfrac{1}{2}\sqrt{t-2\alpha}.$$

(a) Since f and f' are both *proper* fractions, it is evident that $f-f'$ cannot be a whole number, and consequently $p+q+f-f'$ cannot be a whole number, unless $f-f'=0$, or $f=f'$.

(b) For $4x=t+2\alpha$, $\therefore 2\sqrt{x}=\sqrt{t+2\alpha}$, and $\sqrt{x}=\tfrac{1}{2}\sqrt{t+2\alpha}$; in the same manner it may be shewn that $\sqrt{y}=\tfrac{1}{2}\sqrt{t-2\alpha}$.

CASE II.

When A^2-B^2 is not a complete *n*th power.

In this case let C be so assumed as to make $(A^2-B^2)C$ a complete *n*th power, i.e. let $(A^2-B^2)C=\alpha^n$, or $\sqrt[n]{(A^2-B^2)C}=\alpha$; then assume

$$\sqrt[n]{(A+B)\sqrt{C}}=\sqrt{x}+\sqrt{y}, \text{ or } \sqrt[n]{A+B}=\frac{\sqrt{x}+\sqrt{y}}{\sqrt[2n]{C}};$$

$$\therefore \sqrt[n]{(A-B)\sqrt{C}}=\sqrt{x}-\sqrt{y};$$

or $\sqrt[n]{(A^2-B^2)C}\,(\alpha)=x-y$; and,

$$\sqrt[n]{(A^2+B^2+2AB)C}+\sqrt[n]{(A^2+B^2-2AB)C}=2x+2y=t.$$

From which we deduce, as before,

$$\sqrt{x}\pm\sqrt{y}=\tfrac{1}{2}\sqrt{t+2\alpha}\pm\tfrac{1}{2}\sqrt{t-2\alpha};$$

$$\therefore \sqrt[n]{A\pm B}=\frac{\sqrt{x}\pm\sqrt{y}}{\sqrt[2n]{C}}=\frac{\sqrt{t+2\alpha}\pm\sqrt{t-2\alpha}}{2\sqrt[2n]{C}}$$

EXAMPLE 1.

Find the cube root of $26+15\sqrt{3}$.

Here $\left.\begin{array}{l}A=26\\ B=15\sqrt{3}\end{array}\right\}\therefore A^2-B^2=676-675=1$, and $\alpha=1$.

$$\sqrt[3]{A^2+B^2+2AB}=\sqrt[3]{676+675+780\sqrt{3}}=13+f,$$

$$\sqrt[3]{A^2+B^2-2AB}=\sqrt[3]{676+675=780\sqrt{3}}=1-f;$$

$$\therefore t=13+1=14.$$

Hence $\sqrt[3]{A+B}=\frac{1}{2}\sqrt{t+2\alpha}+\frac{1}{2}\sqrt{t-2\alpha}=\frac{1}{2}\sqrt{16}+\frac{1}{2}\sqrt{12}=2+\sqrt{3}$.

Ex. 2.

Find the cube root of $9\sqrt{3}-11\sqrt{2}$.

Here $\left.\begin{array}{l}A=9\sqrt{3}\\ B=11\sqrt{2}\end{array}\right\}\therefore A^2-B^2=243-242=1$, and $\alpha=1$.

$$\sqrt[3]{A^2+B^2+2AB}=\sqrt[3]{243+242+198\sqrt{6}}=9+f,$$

$$\sqrt[3]{A^2+B^2-2AB}=\sqrt[3]{243+242-198\sqrt{6}}=1-f.$$

Hence $t=9+1=10$, and $\frac{1}{2}\sqrt{t+2\alpha}-\frac{1}{2}\sqrt{t-2\alpha}=\frac{1}{2}\sqrt{12}-\frac{1}{2}\sqrt{8}=\sqrt{3}-\sqrt{2}$.

$$\therefore \sqrt[3]{A-B}=\sqrt{3}-\sqrt{2}.$$

Ex. 3.

Find the cube root of $8+4\sqrt{5}$, or $4\sqrt{5}+8$.

Here $\left.\begin{matrix} A=4\sqrt{5} \\ B=8 \end{matrix}\right\}$ $\therefore A^2-B^2=80-64=16$, which is not a *cube* number, and the least number which multiplied into it will produce a cube number is 4,(a) $\therefore C=4$, and $(A^2-B^2)C=16\times 4=64$; hence $\alpha^3=64$, and $\alpha=4$.

Now $\sqrt[3]{(A^2+B^2+2AB)C}=\sqrt[3]{(80+64+64\sqrt{5})4}=10+f$,

$$\sqrt[3]{(A^2+B^2-2AB)C}=\sqrt[3]{(80+64-64\sqrt{5})4}=2-f;$$

$$\therefore t=10+2=12,$$

and $\dfrac{\sqrt{t+2\alpha}+\sqrt{t-2\alpha}}{2\sqrt[2n]{C}}=\dfrac{\sqrt{20}+\sqrt{4}}{2\sqrt[6]{4}}=\dfrac{2\sqrt{5}+2}{2\sqrt[6]{4}}=\dfrac{\sqrt{5}+1}{\sqrt[3]{2}}$.

Hence $\sqrt[3]{4\sqrt{5}+8}=\dfrac{\sqrt{5}+1}{\sqrt[3]{2}}$.

(a) In finding the *least* number by which a given number (a) must be multiplied so as to give a product which shall be a complete nth power, it may be observed, that if a be a *prime* number, it must always be multiplied by a^{n-1}; thus, there is no other number by which 3 can be multiplied to make it a *cube* number, but 3^2 or 9, which gives the product 27; nor is there any other number by which 5 can be multiplied to make it a *biquadrate* number, but 5^3 or 125, which gives the product 625. But if the given number is resolvable into factors, one or more of which are *square, cube,* &c. numbers, then a *less* number than a^{n-1} will answer the purpose. Thus $12=3\times 4=3\times 2^2$; and if 3×2^2 be multiplied by $3^2\times 2$, it gives $3^3\times 2^3$, which is the cube of 3×2; i.e if 12 be multiplied by 18 it gives 216 the cube of 6. Or in general, if the given number (a) be resolvable into factors α, β, γ, &c. such that $a=\alpha^m\beta^p\gamma^q$ &c., then if this number be multiplied by $\alpha^{n-m}\beta^{n-p}\gamma^{n-q}$ &c. it gives $\alpha^n\beta^n\gamma^n$ &c. which is the nth power of $\alpha\beta\gamma$ &c. Thus $360=8\times 9\times 5=2^3\times 3^2\times 5$; here $m=3$, $p=2$, $q=1$; and if it be required to find a multiplier which should make it a *biquadrate* number, then $n=4$, $\therefore n-m=1$, $n-p=2$, $n-q=3$; hence the multiplier is $2\times 3^2\times 5^3=2250$, and we have $360\times 2250=810000$, which is the fourth power of $2\times 3\times 5$ or 30. If one or more of the indices m, p, q, &c. be *greater* than n, then, in finding the multiplier, such multiples of n must be taken as to make the indices of all the factors in the multiplier *positive*; thus if m be greater than n but less than $2n$, then the multiplier to be taken is $\alpha^{2n-m}\beta^{n-p}\gamma^{n-q}$, which gives for the product of it and $\alpha^m\beta^p\gamma^q$ the quantity $\alpha^{2n}\beta^n\gamma^n$, which is the nth power of $\alpha^2\beta\gamma$.

LII.

On the Method of reverting a Series.

Let $x=ay+by^2+cy^3+dy^4+\&c.$, where the value of x is expressed in a series containing the powers of y; by the *reversion* of the series is meant such an operation as shall exhibit the value of y in a series containing the powers of x.

173. Previously to the reversion of a series, it will be necessary to shew the manner in which it may be raised to any power (n). This is done by separating the first term from the rest, and then applying the binomial theorem to the involution of the series so transformed; thus

$$(ax+bx^2+cx^3+dx^4+\&c.)^n=\overline{ax+(bx^2+cx^3+dx^4+\&c.)}|^n$$

$$=a^nx^n+na^{n-1}x^{n-1}(bx^2+cx^3+dx^4+\&c.)+\frac{n(n-1)}{2}a^{n-2}x^{n-2}(bx^2+cx^3+\&c.)^2$$
$$+\frac{n(n-1)(n-2)}{2.3}a^{n-3}x^{n-3}(bx^2+\&c.)^3+\&c.$$

$$=a^nx^n+na^{n-1}x^{n-1}(bx^2+cx^3+dx^4+\&c.)+\frac{n(n-1)}{2}a^{n-2}x^{n-2}(b^2x^4+2bcx^5+\&c.)$$
$$+\frac{n(n-1)(n-2)}{2.3}a^{n-3}x^{n-3}(b^3x^6+\&c.)+\&c.$$

$$=a^nx^n+na^{n-1}bx^{n+1}+\left.\begin{array}{l}na^{n-1}c\\+\frac{n(n-1)}{2}a^{n-2}b^2\end{array}\right\}x^{n+2}+\left.\begin{array}{l}na^{n-1}d\\+n(n-1)a^{n-2}bc\\+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3\end{array}\right\}x^{n+3}+\&c.$$

174. Let us now suppose the following equation to be true, *whatever* be the value of x,

viz. $ax+bx^2+cx^3+dx^4+\&c.=\alpha x+\beta x^2+\gamma x^3+\delta x^4+\&c.$

then, by transposition, we have

$(a-\alpha)x+(b-\beta)x^2+(c-\gamma)x^3+(d-\delta)x^4+\&c.=0\ (B).$

Now whatever is true in the original equation, must also be true in the transposed equation; but it has already been proved with respect to the former equation (Note(b), p.178), that $a=\alpha$; $b=\beta$; $c=\gamma$; $d=\delta$; &c.; hence $a-\alpha=0$; $b-\beta=0$; $c-\gamma=0$; $d-\delta=0$; &c.; from which it follows, that if an equation of the form (B) be true for *any* value of x, its coefficients will all become equal to 0 at the same time.

175. Resuming the equation $x=ay+by^2+cy^3+dy^4+\&c.$

let it be required to find the value of y in terms of x. Transpose x to the other side of the equation, then $ay+by^2+cy^3+dy^4+\&c.-x=0$. Assume $y=\alpha x+\beta x^2+\gamma x^3+\delta x^4+\&c.$; and finding the value of the successive powers of y, by Art. 173, we have

$$\left.\begin{array}{llllll}
ay & = a\alpha x + & a\beta x^2 + & a\gamma x^3 + & a\delta x^4 + \&c. & \\
by^2 & = & b\alpha^2x^2 + & 2b\alpha\beta x^3 + & 2b\alpha\gamma x^4 + \&c. & \\
 & & & + & b\beta^2x^4 + \&c. & \\
cy^3 & = & \ldots & c\alpha^3x^3 + & 3c\alpha^2\beta x^4 + \&c. & \\
dy^4 & = & \ldots & \ldots & d\alpha^4x^4 + \&c. & \\
\&c. & = & \ldots & \ldots & + \&c. & \\
-x & = -x & & & &
\end{array}\right\}=0.$$

Hence, by Art. 174, $a\alpha-1=0$, or $\alpha=\frac{1}{a}$;

$$a\beta+b\alpha^2=0,\text{ or }\beta=-\frac{b\alpha^2}{a}=-\frac{b}{a^3};$$

$$a\gamma+2b\alpha\beta+c\alpha^3=0,\text{ or }\gamma=\frac{-2b\alpha\beta-c\alpha^3}{a}=\frac{2b^2-ac}{a^5};$$

$$a\delta+2b\alpha\gamma+b\beta^2+3c\alpha^2\beta+d\alpha^4=0,\text{ or }\delta=\frac{-2b\alpha\gamma-b\beta^2-3c\alpha^2\beta-d\alpha^4}{a}$$

$$=\frac{-5b^3+5abc-a^2d}{a^7};$$

$$\&c.=\&c.$$

Substitute these values for $\alpha, \beta, \gamma, \delta$, &c. then

$$y=\frac{x}{a}-\frac{bx^2}{a^3}+\frac{(2b^2-ac)x^3}{a^5}-\frac{(5b^3-5abc+a^2d)x^4}{a^7}+\&c.$$

and if $a=1$, or $x=y+by^2+cy^3+dy^4+\&c.$ then

$$y=x-bx^2+(2b^2-c)x^3-(5b^3-5bc+d)x^4+\&c.\ (A).$$

176. In the following chapter it will be shewn, that if l be the logarithm of the number $1+n$, $l=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.$; suppose therefore it was required to find the *number* in terms of the *logarithm*, i.e. to find n in terms of l, then, comparing the equation $l=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.$ with the equation $x=y+by^2+cy^3+dy^4+\&c.$ and substituting l for x and n for y in the equation (A), we should have

$$n=l-bl^2+(2b^2-c)l^3-(5b^3-5bc+d)\,l^4+\&c.$$

where $b=-\frac{1}{2}$, $c=\frac{1}{3}$, $d=-\frac{1}{4}$; &c.;

Hence $-b=\frac{1}{2}$,

$$2b^2-c=\frac{1}{2}-\frac{1}{3}=\frac{1}{2.3}$$

$$-(5b^3-5bc+d)=\frac{5}{8}-\frac{5}{6}+\frac{1}{4}=\frac{15-20+6}{24}=\frac{1}{2.3.4},$$

&c.=&c.

$$\therefore n=l+\frac{l^2}{2}+\frac{l^3}{2.3}+\frac{l^4}{2.3.4}, \text{ \&c.};$$

$$\text{and } 1+n=1+l+\frac{l^2}{2}+\frac{l^3}{2.3}+\frac{l^4}{2.3.4}+\text{ \&c.}$$

CHAP. XI.

ON LOGARITHMS, AND SUBJECTS CONNECTED WITH THEM.

LIII.

Definition and Properties of Logarithms.

177. In the two following series of quantities, $a^x, a^{x'}, a^{x''}, a^{x'''}$, &c. (A); x, x', x'', x''', &c. (B); where a is some given number, and x, x', x'', x''', &c. any variable quantities whatever, the several terms of the series (B) are called the *logarithms* of the several terms corresponding to them in the series (A). Thus, if $a^x=y$, $a^{x'}=y'$, $a^{x''}=y''$, &c. then $x=\log. y$; $x'=\log. y'$; $x''=\log. y''$; &c.

178. In adapting the series (A) to the numbers 1, 2, 3, 4, 5, 6, &c. the given number a must be *greater* than unity, the

first index x must be equal to 0, and the several indices x', x'', x''', &c. must keep continually increasing. For in this case, since (by Art. 66.) $a^0=1$, this series will increase from 1 to infinity; and by properly adjusting the values of x', x'', x''', &c. it is evident that the several quantities $a^{x'}$, $a^{x''}$, $a^{x'''}$, &c. may be made to coincide with the numbers 2, 3, 4, 5, 6, &c. For instance, let $a=10$; then (since $10^0=1$ and $10^1=10$), the indices of 10 which would give $10^{x'}$, $10^{x''}$, $10^{x'''}$, &c. equal to the numbers 2, 3, 4, 5, &c. must be fractions between 0 and 1.

Take for example the number 5. Now $10^{\frac{2}{3}}=\sqrt[3]{10^2}$ $=\sqrt[3]{100}=4.64$; from which we infer, that a fraction (x') somewhat greater than $\frac{2}{3}$ ($=.666666$, &c.) being made the index of 10, would give $10^{x'}=5$; this fraction is found by calculation to be .6989700 very nearly; hence $10^{.6989700}\doteq 5$; i.e. when $a=10$, the logarithm of 5 is .6989700.

179. From hence it appears that the logarithm of any given number will depend upon the value of a, and that different systems of logarithms would be formed by assuming it equal to different numbers, but that (since $a^0=1$) in every system the logarithm of *one* would be 0. This constant quantity a, from whose powers the natural numbers are formed, is called the *base* of the system to which it belongs. But before we proceed to calculate a system of logarithms, it will be proper to explain some of their *properties*.

180. Let N and n be any two numbers belonging to the series (A); let N (for instance) $=a^x$, and $n=a^{x''''}$; then $Nn=a^x\times a^{x''''}=a^{x+x''''}$; but by Art. 177, the *logarithm* of $a^{x+x''''}$ is $x+x''''$, $\therefore$ the logarithm of $Nn=x+x''''=\log. a^x+\log. a^{x''''}=\log. N+\log. n$. In the same manner, if n, n', n'', n''', &c. be any set of numbers belonging to the series (A), it might be shewn that the logarithm of $nn'n''n'''$, &c. $=\log. n+\log. n'+\log. n''+\log. n'''+$&c.; i.e. "the logarithm of the *product* of any number of factors is equal "to the *sum of their logarithms*."

181. Again, $\frac{N}{n} = \frac{a^x}{a^{x''''}} = a^{x-x''''}$; but the logarithm of $\alpha^{x-x''''}$ $= x - x''''$; $\therefore$ the logarithm of $\frac{N}{n} = x - x'''' = \log. a^x - \log. a^x$ $= \log. N - \log. n$; from hence it appears that "the logarithm of the *quotient* of any two numbers is equal to the "*difference of their logarithms;* and that the logarithm of "a *fraction* $\left(\frac{N}{n}\right)$ is equal to the logarithm of its *numerator* "minus the logarithm of its *denominator.*" If N be less than n, then $\log. N - \log. n$ is *negative;* consequently the logarithms of all *proper* fractions are negative quantities.

182. Let $N = a^x$ be raised to the mth power, then $N^m = a^{mx}$; but the logarithm of $a^{mx} = mx$; hence the logarithm of $N^m = mx = m . \log. a^x = m . \log. N$; for the same reason, since $\sqrt[m]{N} = N^{\frac{1}{m}} = a^{\frac{x}{m}}$ the logarithm of $\sqrt[m]{N} = \frac{x}{m}$ $= \frac{\log. N}{m}$; from which we infer that "the logarithm of the "mth power of any number is found by *multiplying* its "logarithm by m; and of the mth root of any number, by "*dividing* its logarithm by m."

183. If the series (A) consists of quantities of the form $a^x, a^{2x}, a^{3x}, a^{4x}$, &c......$a^{nx}$, then the corresponding terms of the series (B) are $x, 2x, 3x, 4x$, &c.....nx; i.e. "if a series "of quantities be in *geometrical* progression, their logarithms will be in *arithmetical* progression."

LIV.

On the Method of finding the Logarithm of any given Number.

184. Let $1 + n$ be any number in the common arithmetical scale, and x its logarithm, then, Art. 177, $a^x = 1 + n$; and let $a = 1 + b$; then, to find the logarithm of $1 + n$, we have only to solve the equation $(1 + b)^x = 1 + n$, where x is the unknown quantity.

Let both sides of this equation be raised to the power h,

then $$(1+b)^{hx}=(1+n)^{h}, \text{ or}$$

$$1+hxb+\frac{hx(hx-1)}{2}b^2+\frac{hx(hx-1)(hx-2)}{2.3}b^3+\&c.=1+hn+\frac{h(h-1)}{2}n^2+\frac{h(h-1)(h-2)}{2.3}n^3+\&c$$

rejecting 1 from each side of the equation and dividing by h, we have

$$x\left(b+\frac{hx-1}{2}b^2+\frac{(hx-1)(hx-2)}{2.3}b^3+\&c.\right)=n+\frac{h-1}{2}n^2+\frac{(h-1)(h-2)}{2.3}n^3+\&c.$$

Now let $h=0$, and we have

$$x(b-\tfrac{1}{2}b^2+\tfrac{1}{3}b^3-\tfrac{1}{4}b^4+\&c.)=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\&c.$$

$$\text{or } x=\log.(1+n)=\frac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.}{b-\frac{1}{2}b^2+\frac{1}{3}b^3-\frac{1}{4}b^4+\&c.}$$

$$=\frac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}$$

$$=M(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\&c.), \text{ if we make}$$

$$\frac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.} \text{ equal to } M.$$

185. But the series which thus expresses the value of x in terms of n, will either diverge, or not converge so quickly as to make the summation of a few terms of it a sufficient approximation to that value; unless n be a *fraction* of a proper degree of smallness. Let, therefore, $n=\frac{1}{N-1}$, where N may be any number greater than 2, then

$$\frac{1+n}{1-n}=\frac{1+\frac{1}{N-1}}{1-\frac{1}{N-1}}=\frac{N}{N-2}$$

and $\log.(1+n)-\log.(1-n)=\log.N-\log.(N-2)$.

Now $\log.(1+n)=M(n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\frac{1}{5}n^5-\&c.)$

and $\left(\begin{matrix}\text{substituting}\\ -n \text{ for } n,\end{matrix}\right)\log.(1-n)=M(-n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4-\frac{1}{5}n^5-\&c.)$

Hence, by subtraction, $\log.(1+n)-\log.(1-n)=2M(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\&c.)$

or $\log. N-\log.(N-2)=2M\left(\frac{1}{N-1}+\frac{1}{3(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right)$

from which we have

$$\log. N=2M\left(\frac{1}{N-1}+\frac{1}{3(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right)+\log.(N-2)$$

which is a very commodious series for constructing a *table* of logarithms, when some value has been assigned to M.

LV.

On the Method of Constructing Logarithmic Tables.

186. Since a may be arbitrarily assumed, let us first suppose it to be such that $\frac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}$ (or M) $=1$; in which case the equation in the foregoing Article becomes

$$\log. N=2\left(\frac{1}{N-1}+\frac{1}{3(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right)+\log.(N-2).$$

But since N must be some number greater than 2, we must find the logarithm of 2, before we can proceed to the actual calculation of a table of logarithms. Now this may be done by making $N=4$ in the first instance, for then we have

$$\log. 4=\log. 2^2=2\log. 2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\&c.\right)+\log. 2,$$

and by subtracting log. 2 from each side of the equation, we have

$$\log. 2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\&c.\text{ to 7 terms}\right)=0.6931472.$$

Having thus obtained the logarithm of 2, we are enabled to construct a *Table* of logarithms, by substituting in the foregoing series all the *prime* numbers for N in succession, and availing ourselves of the *properties* of logarithms for finding the logarithms of all other numbers. Thus, log.

$1 = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 0.0000000$

$2 = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 0.6931472$

$3 = 2\left(\frac{1}{2} + \frac{1}{3.2^3} + \frac{2}{5.2^5} + \text{\&c. to 10 terms}\right) + \log.1(0) = 1.0986123$

$4 = 2 \log.\ 2. \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots = 1.3862944$

$5 = 2\left(\frac{1}{4} + \frac{1}{3.4^3} + \frac{1}{5.4^5} + \text{\&c. to 6 terms}\right) + \log.\ 3 = 1.6094379$

$6 = \log.\ 3 + \log.\ 2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots = 1.7917595$

$7 = 2\left(\frac{1}{6} + \frac{1}{3.6^3} + \frac{1}{5.6^5} + \frac{1}{7.6^7}\right) + \log.\ 5 \dots\dots\dots = 1.9459101$

$8 = \log.\ 1 + \log.\ 2$, or $\log.\ 2^3 = 3 \log.\ 2 \dots\dots\dots = 2.0794415$

$9 = \log.\ 3^2 = 2 \log.\ 3 \dots\dots\dots\dots\dots\dots\dots\dots\dots = 2.1972246$

$10 = \log.\ 5 + \log.\ 2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots = 2.3025851$

&c. = &c. . . . . = &c.

A sufficient number of terms has here been made use of to make the logarithms true to 7 places of decimals. This particular system of logarithms (viz. where $M=1$) are called *Napier's* logarithms, from their inventor; and they are also called *Hyperbolic* logarithms, from their connection with the quadrature of the equilateral hyperbola.

187. To find the *base* of this system of logarithms, let log. $(1+n)=l$, then (since $M=1$), $l = n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \text{\&c.}$, and reverting the series by Art. 176, we obtain

$$1 + n = 1 + l + \frac{l^2}{2} + \frac{l^3}{2.3} + \frac{l^4}{2.3.4} + \text{\&c.}$$

but since $a^1 = a$, the base of any system of logarithms is that number *whose logarithm is* 1; if therefore in this series, which expresses the value of the number in terms of the logarithm, we substitute 1 for l, we shall immediately obtain, for the base of this particular system, the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \text{\&c.}$$

$= 2.7182818$, by actual calculation.

The constant multiplier M is called the *Modulus*; hence, in that particular system of logarithms whose Modulus is 1, the *base* is 2.7182818. Call this number e, and the logarithms of the several powers of e (viz. e, e^2, e^3, e^4, &c.) being 1, 2, 3, 4, &c. we might have interposed in the preceding Table

Log. 2.7182818 $= 1.0000000$

Log. 7.3890559 (being the *square* of 2.7182818) $= 2.0000000$

&c. $=$ &c.

The numbers whose logarithms are 1, 2, 3, 4, &c. in *this* system are, therefore, *decimal* numbers.

188. In the *common* system of logarithms, which are much more convenient for ordinary arithmetical operations than the *Napierian* or *Hyperbolic* logarithms, the base $a = 10$; hence $a^2 = 100$, $a^3 = 1000$, $a^4 = 10000$, &c., and the numbers whose logarithms are 1, 2, 3, 4, &c., in this system, are 10, 100, 1000, 10000, &c. To find the logarithms of the *intermediate* numbers, i.e. to construct a table of logarithms of this kind, we must find the value of M when $a = 10$. Which is done thus,

In a system whose Modulus is M, $\log.(1+n) = M(n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&c.)$

In the Napierian system, $\log.(1+n) = n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&c.$

Hence $\log.(1+n)$ to Modulus $M = M \times$ Nap. $\log.(1+n)$.

In the *common* system, let $1 + n = 10$, then

$$\log.\ 10 = M \times \text{Nap. } \log.\ 10$$

$$\text{or } 1 = M \times 2.3025851, \text{see Art. 186.}$$

$$\therefore M = \frac{1}{2.3025851} = .43429448.$$

For the actual construction of a Table of common logarithms, we must therefore substitute this value of M in the equation at the end of Art. 185, which then becomes

$$\text{Log.}N = .86858896\left(\frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \&c.\right) + \log.(N-2);$$

and it is by the substitution of all the *prime* Numbers in succession for N in this expression, that the following Table is calculated.

Log

$2 = .86858896\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \text{\&c. to 7 terms}\right)^{(a)} = 0.3010300$

$3 = .86858896\left(\frac{1}{3} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \text{\&c. to 10 terms}\right) = 0.4771213$

$4 = 2 \log. 2 \ldots\ldots = 0.6020600$

$5 = \log. \frac{10}{2} = \log. 10 - \log. 2 = 1 - \log. 2 \ \ldots = 0.6989700$

$6 = \log. 3 + \log. 2 \ldots\ldots 0.7781513$

$7 = .86858896\left(\frac{1}{6} + \frac{1}{3.6^3} + \frac{1}{5.6^5} + \frac{1}{7.6^7}\right) + \log. 5 \ldots = 0.8450980$

$8 = \log. 2^3 = 3 \log. 2 \ldots\ldots = 0.9030900$

$9 = \log. 3^2 = 2 \log. 3 \ldots\ldots = 0.9542425$

$10 = \ldots\ldots = 1.0000000$

$11 = .86858896\left(\frac{1}{10} + \frac{1}{3.10^3} + \frac{1}{5.10^5}\right) + \log. 9 \ldots = 1.0413927$

$12 = \log. 3 + \log. 4 \ldots\ldots = 1.0791812$

$13 = .86858896\left(\frac{1}{12} + \frac{1}{3.12^3} + \frac{1}{5.12^5}\right) + \log. 11 \ldots = 1.1139434$

$14 = \log. 7 + \log. 2 \ldots\ldots = 1.1461280$

$15 = \log. 5 + \log. 3 \ldots\ldots = 1.1760913$

$16 = \log. 4^2 = 2 \log. 4 \ldots\ldots = 1.2041200$

$17 = .86858896\left(\frac{1}{16} + \frac{1}{3.16^3} + \frac{1}{5.16^5}\right) + \log. 15 \ldots = 1.2304489$

$18 = \log. 9 + \log. 2 \ldots\ldots = 1.2552725$

$19 = .86858896\left(\frac{1}{18} + \frac{1}{3.18^3} + \frac{1}{5.18^5}\right) + \log. 17 \ldots = 1.2787536$

$20 = \log. 10 + \log. 2 \ldots\ldots = 1.3010300$

$21 = \log. 7 + \log. 3 \ldots\ldots = 1.3222193$

$22 = \log. 11 + \log. 2 \ldots\ldots = 1.3424227$

$23 = .86858896\left(\frac{1}{22} + \frac{1}{3.22^3} + \frac{1}{5.22^5}\right) + \log. 21 \ldots = 1.3617278$

The next number which requires calculation by means of the series, is 29; and from this number to 400 *inclusive, two* terms of the series are sufficient to make the logarithms true to 7 places of decimals. After 400, *one* term is sufficient; thus

---

(a) See Art. 186.

$\log. 401 = \frac{.86858896}{400} + \log. 399 = .0021714724 + 2.6009729$ $= 2.6031444$ (very nearly); and in this manner the table might be continued with great facility to any extent, by means of the logarithms previously calculated. For the most expeditious manner of dividing the number .86858896 by the denominators of the several fractions composing the series, and for the manner of using logarithmic tables, the reader is referred to the Preface annexed to Dr. HUTTON's *Tables*.

189. Since log. 1 = 0, log. 10 = 1, log. 100 = 2, log. 1000 = 3, &c., it follows that the logarithms of all numbers between 1 and 10 will be some decimal number less than unity; between 10 and 100, some decimal number between 1 and 2; between 100 and 1000, some decimal number between 2 and 3; &c. &c. The *whole number* annexed to the decimal is called the *index* or *characteristic* of the logarithm; and consequently for all numbers between 10 and 100, the index is 1; between 100 and 1000, the index is 2; between 1000 and 10000, the index is 3; &c. &c. From the circumstance of log. 10 = 1, it also follows that the logarithms of all numbers in *decuple* proportion involve the same decimal number, and differ only by their *index*.

Thus, Log. 1132 . . . . = 3.0538464.

$$\text{Log. } 113.2 = \log. \frac{1132}{10} = \log. 1132 - 1 = 2.0538464.$$

$$\text{Log. } 11.32 = \log. \frac{113.2}{10} = \log. 113.2 - 1 = 1.0538464.$$

$$\text{Log. } 1.132 = \log. \frac{11.32}{10} = \log. 11.32 - 1 = 0.0538464.$$

$$\text{Log. } .1132 = \log. \frac{1.132}{10} = \log. 1.132 - 1 = \bar{1}.0538464.$$

$$\text{Log. } .01132 = \log. \frac{.1132}{10} = \log. .1132 - 1 = \bar{2}.0538464.$$

$$\text{Log. } .001132 = \log. \frac{.01132}{10} = \log. .01132 - 1 = \bar{3}.0538464^{(a)};$$

where

(*) The index of a logarithm may in all cases be determined by the following simple Rules;—

If

where the negative sign is placed *above* the index of the last three logarithms, to shew that it does not extend to the decimals, which are supposed positive. Thus $\bar{3}.0538464$ means $-3+.0538464$, or -2.9461536.

190. The foregoing property, belonging to that particular system of logarithms arising out of the supposition of the base $a=10$, is not only of great practical utility in their application to arithmetical purposes, but also very much facilitates the construction and use of the tables founded upon that system. Since the same decimal logarithm always applies to a number consisting of the same digits, it follows that, in the construction of a table of common logarithms, it is only necessary to register the digits of the number and the decimal logarithm in parallel columns; for the *index* of the logarithm may always be determined from the actual value of the number; and, *vice versâ*, the actual value of the number may always be determined from the index of the logarithm. For instance, in the common tables where the logarithms are registered for all numbers consisting of five figures, the decimal logarithm belonging to the number 98637 is .9940399; if this number be a *whole* number, then, since it consists of 5 integral digits, we know that its logarithm is 4.9940399; if a decimal point be placed before the last figure, then the value of the number is 9863.7, which has four integral digits, and therefore its logarithm is 3.9940399; if a decimal point be placed before the last figure but one, then the number is 986.37, and its logarithm 2.9940399; &c. &c. On the other hand, if the logarithm 1.9940399 was given to find the corresponding number, then

---

I. If the number be integral, with or without decimals annexed, the index of the logarithm will be *one* less than the number of digits in the integer.

II. If the number be a proper decimal fraction, the *negative* index will be equal to the place of the first significant digit after the decimal point.

then, since the decimal part of it belongs to the digits 98637, and since from the index of the logarithm we know that the number has two integral digits, the figures 98637 must be pointed 98.637; &c. &c. The utility of this system was so obvious, that the tables for ordinary purposes were founded upon it, very soon after the invention of logarithms.

LVI.

On the application of Logarithms to Complex Arithmetical Operations, and to the solution of Exponential Equations.

191. Logarithms are of considerable use in the ordinary operations of multiplying or dividing one large number by another; but it is in the raising of powers, and the extraction of roots, and in their application to complicated numerical expressions, that their utility most plainly appears.

EXAMPLE 1.

Find the 5th root of 2593.

By Art. 182, the logarithm of the 5th root of 2593 $= \frac{\text{log. } 2593}{5} = \frac{3.4138025}{5} = .6827605 = \text{log. } 4.8168$; $\therefore$ the 5th root of $2593 = 4.8168$.

Ex. 2.

Find the value of the fraction $\frac{2^{20} \times 3^7 \times 2.013}{17 \times 9350}$.

By Art. 181, the logarithm of this fraction is equal to the log. of its numerator *minus* log. of its denominator.

By Arts. 180, 182, log. $2^{20} \times 3^7 \times 2.013 = 20$ log. $2 + 7$ log. $3 +$ log. 2.013.

. . . . and, log. $17 \times 9350 =$ log. $17 +$ log. 9350.

Now $20 \times$ log. $2 = 6.0206000$ . . log. $17 = 1.2304489$.

$7 \times$ log. $3 = 3.3398491$ . . log. $9350 = 3.9708116$.

log. $2.013 = 0.3038438$

By addition $= 9.6642929\,(A)$ $\qquad$ $5.2012605\,(B)$.

Subtract (B) from (A), and we have 4.4630324, which is the logarithm of 29042, the number required.

Ex. 3.

Find the value of $\sqrt{\dfrac{(317)^2 \times \sqrt{3} \times \sqrt[3]{5}}{251}}$.

Call the *numerator* of this fraction (N), and its *denominator* (n);

Then, by Art^s. 181, 182, log. of $\sqrt[5]{\dfrac{N}{n}} = \dfrac{\text{log.}\,N - \text{log.}\,n}{5}$.

$$
\begin{aligned}
\text{Now log.}\,(317)^2 &= 2 \times \text{log.}\,317 = 5.0021186. \\
\text{log.}\ \sqrt{3} &= \tfrac{1}{2} \times \text{log.}\ 3 = 0.2385606. \\
\text{log.}\ \sqrt[3]{5} &= \tfrac{1}{3} \times \text{log.}\ 5 = 0.2329900. \\
&\overline{5.4736692} = \text{log.}\ N. \\
\text{log.}\,251 &= 2.3996737\,; \\
\therefore\ &\overline{3.0739955} = \text{log.}\,N - \text{log.}\,n.
\end{aligned}
$$

Hence $\dfrac{\text{log.}\,N - \text{log.}\,n}{5} = \dfrac{3.0739955}{5} = 0.6147991$, which is the logarithm of 4.119, the number required.

Ex. 4.

Find a fourth proportional to the 6th power of 9, the 4th power of 7, and the 5th power of 5.

Let x = the number required, then $9^6 : 7^4 :: 5^5 : x = \dfrac{7^4 \times 5^5}{9^6}$;

$\therefore$ log. $x = 4$ log. $7 + 5$ log. $5 - 6$ log. $9 = 3.3803920 + 3.4948500 - 5.7254550 = 1.1497870 =$ log. 14.118; hence $x = 14.118$.

192. Equations into which the unknown quantity enters in the form of an *index*, are called *Exponential Equations*; and are solved by means of Logarithms, as in the following examples.

Ex. 5.

Find the value of x in the equation $a^x = b$.

Taking the *logarithm* of the equation $a^x = b$, we have $x.\text{log.}\,a = \text{log.}\,b$, $\therefore x = \dfrac{\text{log.}\,b}{\text{log.}\,a}$; thus, let $a = 5$, $b = 100$, then in the equation $5^x = 100$, $x = \dfrac{\text{log.}\,100}{\text{log.}\,5} = \dfrac{2.0000000}{0.6989700} = 2.861$.

Ex. 6.

To find the value of x in the equation $a^{b^x}=c$.

Assume[a] $b^x=y$, then $a^y=c$, and $y.\log. a=\log. c$, $\therefore y=\frac{\log. c}{\log. a}$; hence $b^x=\frac{\log. c}{\log. a}$ (which let) $=d$. Take the logarithm of the equation $b^x=d$, then (by Ex. 5.) $x=\frac{\log. d}{\log. b}$; thus, let $a=9$, $b=3$, $c=1000$, then in the equation $9^{3^x}=1000$, $\frac{\log. c}{\log. a}=\frac{\log. 1000}{\log. 9}=3.14(=d)$; and $x=\frac{\log. d}{\log. b}=\frac{\log. 3.14}{\log. 3}=\frac{.4969296}{.4771213}=1.04$.

Ex. 7. Find the value of $\frac{31\times 33\times 255\times 315}{35\times 357}$.

Answer, 6576.4.

Ex. 8. Divide the 20th power of 2 by the 12th power of 3.

Answ. 1.973.

Ex. 9. Find a *third proportional* to $\sqrt[5]{117}$ and $\sqrt[3]{137}$.

Answ. 10.252.

Ex. 10. Find the value of $\frac{\sqrt[7]{935\times\sqrt{14}\times\sqrt[3]{100}}}{\sqrt[4]{2}}$.

Answ. 3.3593.

Ex. 11. Find the value of x in the equation $\frac{ab^x+c}{d}=e$.

Answ. $x=\frac{\log.(de-c)-\log. a}{\log. b}$.

LVII.

On the Summation of Geometric Series.

193. Logarithms are found very useful in ascertaining the value of S in the equation $S=\frac{ar^n-a}{r-1}$ or $\frac{a-ar^n}{1-r}$, where n is not a very small number.

Example 1.

Find the sum of 20 terms of the series $1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}$, &c.

---

[a] In considering the nature of an exponential of the form a^{b^x}, it must be recollected that it means *a to the power of* b^x, and not a^b *to the power of* x.

$$\left.\begin{array}{l} \text{Here } a=1 \\ r=\frac{3}{2} \\ n=20 \end{array}\right\} \therefore S=\frac{ar^n-a}{r-1}=\frac{1\times\left(\frac{3}{2}\right)^{20}-1}{\frac{3}{2}-1}=2\times\left(\left(\frac{3}{2}\right)^{20}-1\right).$$

$$\text{Now log. }\left(\frac{3}{2}\right)^{20}=20\times\text{log. }\frac{3}{2}$$

$$=20\times(\text{log. }3-\text{log. }2)$$

$$=3.5218260=\text{log. }3325.263;$$

$$\therefore\ \left(\frac{3}{2}\right)^{20}=3325.263.$$

$$\text{Hence } S=2\times\left(\left(\frac{3}{2}\right)^{20}-1\right)=2\times 3324.263=6648.526.$$

Ex. 2.

Find the sum of 10 terms of the series 1, $\frac{5}{6}$, $\frac{25}{36}$, $\frac{125}{216}$, &c.

$$\left.\begin{array}{l} \text{Here } a=1 \\ r=\frac{5}{6} \\ n=10 \end{array}\right\} \therefore S=\frac{a-ar^n}{1-r}=\frac{1-1\times\left(\frac{5}{6}\right)^{10}}{1-\frac{5}{6}}=6\times\left(1-\left(\frac{5}{6}\right)^{10}\right).$$

$$\text{Now log. }\left(\frac{5}{6}\right)^{10}=10\times\text{log. }\frac{5}{6}.$$

$$=10\times(\text{log. }5-\text{log. }6.)$$

$$=10\times -.0791813,$$

$$=-.7918130.$$

$$=.2081870-1.0000000.$$

$$=\text{log. }1.6150-\text{log. }10.$$

$$\therefore\ \left(\frac{5}{6}\right)^{10}\quad 10 \quad =.1615.$$

$$\text{Hence } S=6\left(1-\left(\frac{5}{6}\right)^{10}\right)=6(1-.1615)=5.031.$$

194. If the *sum* of the series, the *common ratio*, and the *first term* be given; the *number of terms* may be found thus (See Art. 111);

$$\text{Since } rS-S=ar^n-a;$$

$$\text{By } \textit{transposition}, \ ar^n=rS-S+a,$$

$$\text{and } r^n=\frac{rS-S+a}{a};$$

$$\therefore \log. r^n \text{ or } n\times\log. r=\log.(rS-S+a)-\log. a.$$

$$\text{Hence } n=\frac{\log.(rS-S+a)-\log. a}{\log. r}.$$

Ex. 3.

The *sum* of a geometric series is 6560, its *first term* 2, and *common ratio* 3; What is the number of terms?

$$\left.\begin{aligned}\text{Here } S&=6560\\ a&=2\\ r&=3\end{aligned}\right\}\quad \therefore n=\frac{\log.(rS-S+a)-\log.}{\log. r}$$

$$=\frac{\log. 13122-\log. 2}{\log. 3}$$

$$=\frac{3.8169700}{.4771213}=8.$$

Ex. 4. A servant agreed to serve his master for one year (13 months,) at the rate of sixpence for the *first* month, a shilling for the *second*, two shillings for the *third*, and so on; What had he to receive at the end of the year?

ANSWER, 204*l.* 15*s.* 6*d.*

Ex. 5. Find the sum of 11 terms of the series, $1, \frac{5}{4}, \frac{25}{16}$, &c.

ANSW. 42.566.

Ex. 6. The *sum* of a geometric series is 1023, the *first term* 1, and *common ratio* 2; Find the *number of terms*.

ANSW. 10.

Ex. 7. A person undertakes a journey of 364 miles, going *one* mile the *first* day, *three* the *second*, *nine* the *third*, and so on; When will he arrive at his journey's end?

ANSW. In 6 days.

LVIII.

On Compound Interest.

Let (P) be the *principal*, or sum put out to *compound* interest; (r) the fraction which expresses the *rate* of interest per cent.[a]; (A) the *amount* at the end of (n) years, the interest being paid yearly; Then the following Theorems may be established, by means of logarithms.

THEOREM 1.

195. "$Log.\ A = log.\ P + n \times log.\ (1+r)$."

For since £.1, at the end of the *first* year, becomes $1+r$, and that the amount is increased *each* year in the same ratio, we have, by the rule of proportion,

$1 : 1+r :: P \quad : P(1+r) =$ amount of P at end of *first year*.
$1 : 1+r :: P(1+r) : P(1+r)^2 =$ . . . . *second year*.
$1 : 1+r .: P(1+r)^2 : P(1+r)^3 =$ . . . . *third year*.
&c. &c.

So that, at the end of n years, the amount is $P(1+r)^n$.

$$\text{Hence } A = P(1+r)^n;$$

and, taking the *logarithm*, $\log.\ A = \log.\ P + n \times \log.\ (1+r)$.

From thence we deduce

$$\text{Log.}\ P = \log.\ A - n \times \log.(1+r).$$

$$\text{Log.}\ 1+r) = \frac{\log.\ A - \log.\ P}{n};$$

$$\text{and } n = \frac{\log.\ A - \log.\ P}{\log.\ (1+r)}.$$

Any *three* of the quantities A, P, r, n, being given, the *fourth* may therefore be found.

THEOR. 2.

196. "Let $A = m\ P$, then $n = \dfrac{\log.\ m}{\log.(1+r)}$."

For, in this case, $mP = P(1+r)^n$.

Divide by P, then $m = (1+r)^n$.

---

[a] That is, the fraction which expresses the ratio of the interest to the principal. Let the interest, for example, be 5 per cent.; then this fraction (r) will be $\frac{5}{100}$ or $\frac{1}{20}$.

Take the logarithm, $\log.m = n \times \log.(1+r)$; $\therefore n = \frac{\log.\, m}{\log.(1+r)}$.

By means of this Theorem, we ascertain the period or number of years in which a sum of money would *double*, *treble*, &c. or amount to m times itself, when put out at compound interest, at r rate per cent.

THEOR. 3.

197. "*Suppose the interest to be paid half yearly, and* "*at the same time converted into principal, then will* "$\log.\ A = \log.\ P. + 2n \times \log.(1 + \frac{1}{2}r)$."

For in this case, $2n$ must be substituted for n, and $\frac{1}{2}r$ for r. Hence, at the end of n years, $A = P(1 + \frac{1}{2}r)^{2n}$;
and, taking the logarithm, $\log.A = \log.P + 2n \times \log.(1 + \frac{1}{2}r)$.

THEOR. 4.

198. "*Suppose now, that besides the interest being con-* "*verted into principal at the end of every year, the sum P is* "*at the same time invested in capital; then the amount* (A), "*at the end of* n *years, will be* $\frac{PR(R^n - 1)}{R-1}$ (if $R = 1 + r$."

In this case, the principal (P) is put out for $n, n-1, n-2$, &c. years, in succession; the amount therefore is the sum of the several amounts of (P) put out for $n, n-1, n-2$, &c. years;

$$A = P(1+r)^n + P(1+r)^{n-1} + P(1+r)^{n-2} + \&c. + P(1+r).$$
$$= (\text{if } 1+r=R)\, PR^n + PR^{n-1} + PR^{n-2} + \&c. \ \ldots + PR.$$
$$= P(R^n + R^{n-1} + R^{n-2}\,\&c. \ \ldots\ldots + R).$$
$$= P \times (\text{Geo.Prog. } \textit{first term } R, \textit{common ratio } R) = \frac{P(R^{n+1} - R)}{R-1}.$$
$$= \frac{PR(R^n - 1)}{R-1}.$$

EXAMPLE 1.

What would be the amount of 200*l.* placed out for 7 years, at 4 per cent. compound interest?

Here $P = 200$
$r = \frac{1}{25}$
$1 + r = 1 + \frac{1}{2}$
$= 1.04$
$n = 7$;

$\therefore$ by TH. 1. $\log.A = \log.P + n \times \log.(1+r)$.
$= \log.\ 200 + 7 \times \log.\ 1.04$.
$= 2.4202631$.
$= \log.\ 263.18$.
$= \log.\ 263.18$.
Hence, $A = 263l.\ 3s.\ 7\frac{1}{5}d.$

Ex. 2.

How much money must be placed out at compound interest, to amount to 500*l.* in 12 years, at 5 per cent.?

$$\left.\begin{aligned} \text{Here } A &= 500 \\ r &= \frac{1}{20} \\ 1+r &= 1+\frac{1}{20} \\ &= 1.05 \\ n &= 12. \end{aligned}\right\} \quad \begin{aligned} \text{By Th. 1. } \log. P &= \log. A - n \times \log.(1+r). \\ &= \log.500 - 12 \times \log.1.05. \\ &= 2.4446984. \\ &= \log. 278.418. \\ \text{Hence, } P &= 278l.\ 8s.\ 4\tfrac{3}{10}\,d. \end{aligned}$$

Ex. 3.

At what rate of interest must 400*l.* be placed out, that it may amount to 569*l.* 6*s.* 8*d.* in 9 years, at compound interest?

$$\left.\begin{aligned} \text{Here } A &= \overset{\pounds\ \ s\ \ d.}{569.6.8.} \\ P &= 400, \\ n &= 9. \end{aligned}\right\} \quad \begin{aligned} \text{By Th. 1. } \log.(1+r) &= \frac{\log. A - \log. P}{n}. \\ &= \frac{\log. 569.33 - \log. 400}{9}. \\ &= .0170338. \\ &= \log. 1.04 = \log.\left(1+\frac{1}{25}\right). \end{aligned}$$

$$\text{Hence } 1+r = 1+\frac{1}{25};$$

$$\therefore\ r = \frac{1}{25}, \text{ or the rate of interest is 4 per cent.}$$

Ex. 4.

In how many years will 500*l.* amount to 900*l.*, at 5 per cent. compound interest?

$$\left.\begin{aligned} \text{Here } A &= 900 \\ P &= 500 \\ r &= \frac{1}{20} \\ 1+r &= 1.05. \end{aligned}\right\} \quad \begin{aligned} \text{By Theor. 1. } n &= \frac{\log. A - \log. P}{\log.(1+r)}. \\ &= \frac{\log. 900 - \log. 500}{\log. 1.05.} \\ &= \frac{.2552725}{.0211893} = 12.04 \text{ years.} \end{aligned}$$

Ex. 5.

In what time will a sum of money *double* and *treble* itself, at 5 per cent. compound interest?

By Theor. 2. $\left(\text{since } r=\frac{1}{20}\right)$,

$$\text{If } m=2\text{, then time of } doubling=\frac{\log. 2}{\log. 1.05}=\frac{.3010300}{.0211893}=14.2 \text{ years.}$$

$$m=3 \;\ldots\ldots\; \text{of } trebling=\frac{\log. 3}{\log. 1.05}=\frac{.4771213}{.0211893}=22.5 \text{ years.}$$

Ex. 6.

Supposing the interest to be paid *half yearly*, what will be the amount of 500*l.* in 8 years, at 5 per cent. compound interest?

Here $P=500$

$r=\frac{1}{20}$

$1+\frac{1}{2}r=1.025$

$n=8.$

By Th. 3. $\log. A=\log. P+2n\times\log.(1+\frac{1}{2}r)$.

$=\log. 500+16\times\log.(1.025)$.

$=2.8705524=\log. 742.25.$

Hence $A=742l.\ 5s.$

Ex. 7.

Suppose a person to place out annually 100*l.* for 10 successive years, and suffer the whole to accumulate at the rate of 5 per cent. compound interest; What sum would he have to receive at the end of the tenth year?

$$\left.\begin{aligned}\text{Here } P&=100\\ R&=1.05\\ n&=10;\end{aligned}\right\}\quad \therefore \text{ by Theor. 4,}\quad A=\frac{PR(R^n-1)}{R-1}=\frac{105(\overline{1.05}|^{10}-1)}{.05}.$$

$$=2100(\overline{1.05}\quad {\cdot}1).$$

Now $\log.(1.05)^{10}=10\times\log. 1.05.$

$=.2118930.$

$=\log. 1.6289;\ \therefore\ (1.05)^{10}-1=.6289.$

Hence $A=2100\times.6289.$

$=1320l.\ 13s.\ 9\frac{3}{5}d.$

EXAMPLES FOR PRACTICE.

Ex. 8. What would be the amount of 1000*l.* placed out at compound interest of 5 per cent. for 10 years?

ANSWER. 1628*l.* 18*s.*

Ex. 9. What sum must be placed out at compound interest, at 4 per cent., to amount to 2000*l.* in 15 years?

Answ. 1110*l.* 10*s.*

Ex. 10. At what rate of compound interest must 518*l.* 6*s.* be placed out, to amount to 600*l.* in 3 years?

Answ. 5 per cent.

Ex. 11. In how many years will 200*l.* amount to 318*l.* 16*s.* at 6 per cent. compound interest?

Answ. 8 years.

Ex. 12. In how many years will a sum of money *double* itself, at 4 per cent. compound interest?

Answ. 17.6 years.

Ex. 13. Find the amount of 1200*l.* put out to compound interest at 6 per cent. for 10 years, the interest being converted into principal every *half* year.

Answ. 2167*l.* 6*s.*

Ex. 14. Suppose a person to place out annually the sum of 20*l.* for 40 successive years, and suffer the whole to accumulate, at the rate of 5 per cent. compound interest; What would he have to receive at the end of 40 years?

Answ. 2536*l.* 16*s.*

LIX.

On the Method of finding the Increase of Population in any Country, under given circumstances of Births and Mortality.

199. "Let (P) represent the population of a country at " any given period; $\left(\frac{1}{m}\right)$ the fractional part of the popu-" lation which die in a year (or ratio of mortality); $\left(\frac{1}{b}\right)$ the proportion of births in a year; then, if (A) represents the " state of the population at the end of (n) years, $\log. A =$ " $\log. P + n \times \log. \left(1 + \frac{m-b}{mb}\right)$."

The rate of increase of population in one year $=\frac{1}{b}-\frac{1}{m}=\frac{m-b}{mb}$; $\therefore 1:1+\frac{m-b}{mb}::P:P\left(1+\frac{m-b}{mb}\right)$=state of the population at the end of the *first* year.

But it is increased every year in the same proportion; $1:1+\frac{m-b}{mb}::P\left(1+\frac{m-b}{mb}\right):P\left(1+\frac{m-b}{mb}\right)^2$ =state of the population at the end of the *second* year.

In the same manner we may prove, th[illegible] the state of the population at the end of (n) years will be $P\left(1+\frac{m-b}{mb}\right)^n$.

$$\text{Hence } A=P\left(1+\frac{m-b}{mb}\right)^n;$$

$$\text{and } \log. A=\log. P+n\times\log.\left(1+\frac{m-b}{mb}\right).$$

From which we deduce,

$$\text{Log. } P=\log. A-n\times\log.\left(1+\frac{m-b}{mb}\right).$$

$$n=\frac{\log. A-\log. P}{\log.\left(1+\frac{m-b}{mb}\right)}.$$

$$\log\left(1+\frac{m-b}{mb}\right)=\frac{\log. A-\log. P}{n}.$$

Of the quantities A, P, m, b, n, any *four* being given, the *fifth* may therefore be found.

EXAMPLE 1.

Suppose the population of Great Britain in the year 1800 to have been ten millions; that $\frac{1}{40}$th part *die* annually; that the births are to the deaths as 40 : 30; and that no emigration takes place during the present century; What will be the state of its population in the year 1900?

Here $P=10000000$ Now $\log. A=\log. P+n\times\log.\left(1+\frac{m-b}{mb}\right)$

$n=100$

$m=40$ $\qquad=\log. 10000000+100\times\log.\frac{121}{120}$

$b=30$; and

$\qquad=7.3604200$

$\therefore 1+\frac{m-b}{mb}=\frac{121}{120}$. $\qquad-\log. 22931000.$

Hence $A=22931000$.

Ex. 2.

Suppose the population of France, in the year 1792, to have been 27000000; the *ratio of mortality* during the 18th century to have been $\frac{1}{30}$th, and the *number of births* $\frac{1}{26}$th; What was the state of its population in the year 1700?

Here $A=27000000$, $n=92$, $m=30$, $b=26$

$$\therefore 1+\frac{m-b}{mb}=\frac{196}{195}$$

$$\text{Log.}P=\log.A-n\times\log.\left(1+\frac{m-b}{mb}\right)$$

$$=\log.27000000-92\times\log.\frac{196}{195}$$

$$=7.2269858$$

$$=\log.\ 16864980, \text{ nearly};$$

$$\therefore P=16864980.$$

Ex. 3.

Suppose the population of North America to have been five millions in the year 1800; In how many years will it amount to 16 millions, taking the *ratio of mortality* at $\frac{1}{45}$th, and the annual proportion of *births* at $\frac{1}{24}$th?

Here $A=16000000$, $P=5000000$, $m=15$, $b=21$;

$$+\frac{m-b}{mb}=\frac{367}{360}.$$

$$n=\frac{\log.\ A-\log.\ P}{\log.\left(1+\frac{m-b}{mb}\right)}$$

$$=\frac{\log.\ 16000000-\log.\ 5000000}{\text{og.}\frac{367}{360}}$$

$$=\frac{.5051500}{.0083636}=60.39 \text{ years.}$$

Ex. 4.

The population of a province, in the year 1760, was estimated at 500000 persons; in the year 1800, it amounted to 720000; from the bills of mortality it appeared, that, upon an average, $\frac{1}{50}$th part of the population had *died* annually; no register had been kept of the *births*; What was the annual proportion of *them* during this period?

Here $A=720000$, $P=500000$, $m=50$, $=40.$

$$\text{Log.}\left(1+\frac{m-b}{mb}\right)=\frac{\log.A-\log.P}{n},$$

$$\text{or }\log.\left(1+\frac{50-b}{50b}\right)=\frac{\log.720000-\log.500000}{40}$$

$$=.0039590=\log.\ 1.009.$$

Hence

$$\text{Hence } 1+\frac{50-b}{50b}\quad 1.009=1+\frac{}{1000}$$

$$\text{and } \frac{50-b}{50b}=\frac{9}{1000}.$$

$$\therefore\ 50000-1000b=450b,$$

$$\text{or } b=\frac{50000}{1450}=34.4.$$

The annual proportion of *births*, therefore, was about $\frac{1}{34}$th.

200. But "in any country, under *given* circumstances of "births and mortality, the fraction $\frac{m-b}{mb}$ is always a *given* "quantity; Let it be represented by $\frac{1}{p}$; then the relation "between the four quantities, A, P, p, n, is expressed by "$A=P\left(1+\frac{1}{p}\right)^n$. If $A=mP$, we have $mP=P\left(1+\frac{1}{p}\right)^n$, or "$m=\left(1+\frac{1}{p}\right)^n$; and taking the logarithm, $\log. m=n\times\log.$ "$\left(1+\frac{1}{p}\right)^n$. Hence we deduce the *six* following formulæ."

I. $\text{Log. } A=\log. P+n\log.\left(1+\frac{1}{p}\right).$

II. $\text{Log. } P=\log. A-n\log.\left(1+\frac{1}{p}\right).$

III. $n=\dfrac{\log. A-\log. P}{\log.\left(1+\frac{1}{p}\right)}.$

IV. $\text{Log.}\left(1+\frac{1}{p}\right)=\dfrac{\log. A-\log. P}{n}.$

V. $\dfrac{\log. m}{\log.\left(1+\frac{1}{p}\right)}$, for finding the *period* in which the population would be increased m times.

VI. $\text{Log.}\left(1+\frac{1}{p}\right)=\dfrac{\log. m}{n}$, for finding the rate $\left(\frac{1}{p}\right)$ at which the population would be increased m *times* in n years.

The following Questions are intended to illustrate the use of these formulæ, in the order in which they stand.

QUESTION 1.

Suppose the population of a country to begin with *six* persons, and to increase annually by $\frac{1}{16}$th of the whole; What will be the state of its population, at the end of 200 years? **Answer,** 1106448 persons.

QUESTION [illegible]

If (as stated in the 3d Example) the population of North America was five millions in the year 1800, and the rate of increase has been $\frac{7}{300}$th for 50 years previous; What was the state of its population in the year 1750?

Answ. 1908930 persons.

QUESTION 3.

Suppose the population of an empire to be 40 millions, and the annual increase $\frac{1}{195}$th; How long will it be before it amounts to 50 millions? **Answ.** 43.6 years.

QUESTION 4.

What must be the *rate of increase*, that the population of a country may be changed from 1106400 persons to five millions, in 100 years? **Answ.** About $\frac{1}{66}$th annually.

QUESTION 5.

By means of the formula $n = \dfrac{\log. m}{\log.\left(1 + \frac{1}{p}\right)}$; *verify* the following Table.

$\frac{1}{p}$	Period of *doubling*.	Period of *trebling*.	Period of being increased 10 *times*.
$\frac{1}{120}$	83.5 years	132.3 years	277.4 years.
$\frac{1}{52}$	36.3 years	57.6 years	120.8 years.

QUESTION 6.

What must be the annual increase of population in any country, that it may *double* itself every century?

Answ. Between $\frac{1}{143}$d and $\frac{1}{144}$th.

201. Supposing that a *census* of the whole population of a country is taken every n years, and that it is found to have increased π *per cent.* during that interval, then if P represents the amount of the population at the *commencement* of the n years, $P+\frac{\pi P}{100}$ will represent the amount of the population at the *end* of the n years.

If the *annual* increase be $\frac{1}{p}$, then (by Art. 200) the amount of the population at the end of n years is $P\left(1+\frac{1}{p}\right)^n$; hence

$$P\left(1+\frac{1}{p}\right)^n=P+\frac{\pi P}{100}=P\left(1+\frac{\pi}{100}\right)$$

$$\text{or}\left(1+\frac{1}{p}\right)^n=1+\frac{\pi}{100}=\frac{100+\pi}{100};$$

$$\therefore n.\log.\left(1+\frac{1}{p}\right)=\log.(100+\pi)-\log.100$$

$$=\log.(100+\pi)-2,\text{ since }\log.100=2,$$

$$\text{and }\log.\left(1+\frac{1}{p}\right)=\frac{1}{n}\Big(\log.(100+\pi)-2\Big).$$

Substitute this value of $\log.\left(1+\frac{1}{p}\right)$ in the expression $\frac{\text{Log. } m}{\log.\left(1+\frac{1}{p}\right)}$ (Formula V. Art. 200), and we have $\frac{\text{Log. } m}{\frac{1}{n}\Big(\log.(100+\pi)-2\Big)}$ for the number of years in which the population of a country will be increased m times, if it goes on increasing at the same rate as it has done for the last n years preceding the period at which the *census* is taken.

202. If the census be taken every *ten* years, and the period of *doubling* be required, then $n=10$, $m=2$, and the foregoing expression becomes $\frac{\text{Log. } 2}{\frac{1}{10}\Big(\log.(100+\pi)-2\Big)}$. By substituting in it for π the particular value of the *per centage*, the following Table exhibits the corresponding *period of doubling.*

LX.

A TABLE, exhibiting the Period in which the Population of a Country has a tendency to DOUBLE itself, from an estimate of its increase *per cent.* taken at the end of every Ten Years.

I.	II.	III.
Per Centage Increase in ten years.	Numerical Value of $\frac{1}{10}\left(\log.(100+\pi)-2\right)$.	Period of Doubling. $\dfrac{\text{Log. } 2, \text{ or } .3010300}{\frac{1}{10}\left(\log.(100+\pi)-2\right)}$.
$\pi =$ 1.0	.00043214	696.60 years
1.5	.00064660	465.55
2.0	.00086002	350.02
2.5	.00107239	280.70
3.0	.00128372	234.49
3.5	.00149403	201.48
4.0	.00170333	176.73
4.5	.00191163	157.47
5.0	.00211893	142.06
$\pi =$ 5.5	.00232525	129.46 years
6.0	.00253059	118.95
6.5	.00273496	110.06
7.0	.00293838	102.44
7.5	.00314085	95.84
8.0	.00334238	90.06
8.5	.00354297	84.96
9.0	.00374265	80.43
9.5	.00394141	76.37
10.0	.00413927	72.72
$\pi =$ 10.5	.00433623	69.42 years
11.0	.00453230	66.41
11.5	.00472749	63.67
12.0	.00492180	61.16
12.5	.00511525	58.84
13.0	.00530784	56.71
13.5	.00549959	54.73
14.0	.00569049	52.90
14.5	.00588055	51.19
15.0	.00606978	49.59

A TABLE, exhibiting the Period in which the Population of a Country has a tendency to DOUBLE itself, from an estimate of its increase *per cent.* taken at the end of every Ten Years.

I.	II.	III.
Per-Centage Increase in ten years.	Numerical Value of $\frac{1}{10}\left(\log.(100+\pi)-2\right)$.	Period of Doubling. $\dfrac{\text{Log. 2, or .3010300}}{\frac{1}{10}\left(\log.(100+\pi)-2\right)}$
π = 15.5	.00625820	48.10 years
16.0	.00644580	46.70
16.5	.00663259	45.38
17.0	.00681859	41.14
17.5	.00700379	42.98
18.0	.00718820	41.87
18.5	.00737184	40.83
19.0	.00755470	39.81
19.5	.00773679	38.91
20.0	.00791812	38.01
π = 20.5	.00809870	37.17 years
21.0	.00827854	36.36
21.5	.00845763	35.59
22.0	.00863598	31.85
22.5	.00881361	31.15
23.0	.00899051	33.48
23.5	.00916670	32.83
21.0	.00934217	32.22
24.5	.00951694	31.63
25.0	.00969100	31.06
π = 25:5	.00986437	30.51 years
26.0	.01003705	29.99
26.5	.01020905	29.48
27.0	.01038037	28.99
27.5	.01055102	28.53
28.0	.01072100	28.07
28.5	.01089031	27.61
29.0	.01105897	27.22
29.5	.01122698	26.81
30.0	.01139434	26.41

A TABLE, exhibiting the Period in which the Population of a Country has a tendency to DOUBLE itself, from an estimate of its increase *per cent.* taken at the end of every Ten Years.

I.	II.	III.
Per Centage Increase in ten years.	Numerical Value of $\frac{1}{10}(\log.(100+\pi)-2)$.	Period of Doubling. $\dfrac{\text{Log. } 2, \text{ or } .3010300}{\frac{1}{10}(\log.(100+\pi)-2)}$
$\pi = 30.5$	.01156105	26.03 years
31.0	.01172713	25.67
31.5	.01189258	25.31
32.0	.01205739	24.96
32.5	.01222159	24.63
33.0	.01238516	24.30
33.5	.01254813	23.99
34.0	.01271048	23.68
34.5	.01287223	23.38
35.0	.01303338	23.09
$\pi = 35.5$	.01319393	22.81 years
36.0	.01335389	22.54
36.5	.01351327	22.27
37.0	.01367206	22.01
37.5	.01383027	21.76
38.0	.01398791	21.52
38.5	.01414498	21.28
39.0	.01430148	21.04
39.5	.01445742	20.82
40.0	.01461820	20.59
$\pi = 41$	.01492191	20.17 years
42	.01522883	19.76
43	.01553360	18.37
44	.01583625	19.00
45	.01613680	18.65
46	.01643529	18.31
47	.01673173	17.99
48	.01702617	17.68
49	.01731863	17.38
50	.01760913	17.09

This is the Table of which the *first* and *third* columns have been inserted by Mr. Malthus, at page 498, Vol. I. of the sixth edition of his Essay on Population.

From the Parliamentary Report of the Population of England and Wales, it appears

That in 1800 it amounted to 9168000
1810 10502500
1820 12218500
[persons

which gives an increase of about 14.5 per cent. from 1800 to 1810, and of about 16.3 per cent. from 1810 to 1820.

From hence, by referring to the Table, we infer that, taking the *average* rate of increase from 1800 to 1810, the population of England and Wales had in 1810 a *tendency* to double itself in about 51 years; and, taking the average rate of increase from 1810 to 1820, it had in 1820 a tendency to double itself in about 46 years.

THE END.

Richard Watts, Printer, Crown Court, Temple Bar.

1D3